LOCALLY MAXIMALLY ENTANGLED STATES OF MULTIPART QUANTUM SYSTEMS

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Aside: normally, I explore connections between QI and gravity

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High Energy Physics - Theory

Lectures on Gravity and Entanglement

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The AdS/CFT correspondence provides quantum theories of gravity in which spacetime and gravitational physics emerge from ordinary non-gravitational quantum systems with many degrees of freedom. Recent work in this context has uncovered fascinating connections between quantum information theory and quantum gravity, suggesting that spacetime geometry is directly related to the entanglement structure of the underlying quantum mechanical degrees of freedom and that aspects of spacetime dynamics (gravitation) can be understood from basic quantum information theoretic constraints. In these notes, we provide an elementary introduction to these developments, suitable for readers with some background in general relativity and quantum field theory. The notes are based on lectures given at the CERN Spring School 2014, the Jerusalem Winter School 2014, the TASI Summer School 2015, and the Trieste Spring School 2015.

today: pure QI

Locally Maximally Entangled States:

consider Hilbert space:

$$\mathcal{H} = \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \ldots \otimes \mathcal{H}_{d_n}$$

dimensions

states:

$$|\psi\rangle = \sum_{i} \psi_{i_1 \ldots i_n} |i_1\rangle \otimes \ldots \otimes |i_n\rangle$$

define $\rho_k = \mathrm{Tr}_{\bar{k}} |\psi\rangle\langle\psi|$: reduced density matrix for kth subsystem

Then $\mathcal{H}_{LME} \equiv \left\{ |\psi\rangle \in \mathcal{H} \;\middle|\; \rho_k = \frac{1}{d_k}\mathbb{1} \quad \forall k \right\}$

LME ≡ each elementary subsystem maximally mixed

EXAMPLES:

Bell state: $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle \otimes |i\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d$

GHZ state: $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|000\rangle + |111\rangle \right) \in \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$

various quantum error correcting codes, cluster states, perfect tensors, ...

- appear in many applications

THE SPACE $\mathcal{H}_{LME}/K$

LME property preserved under local change of basis:

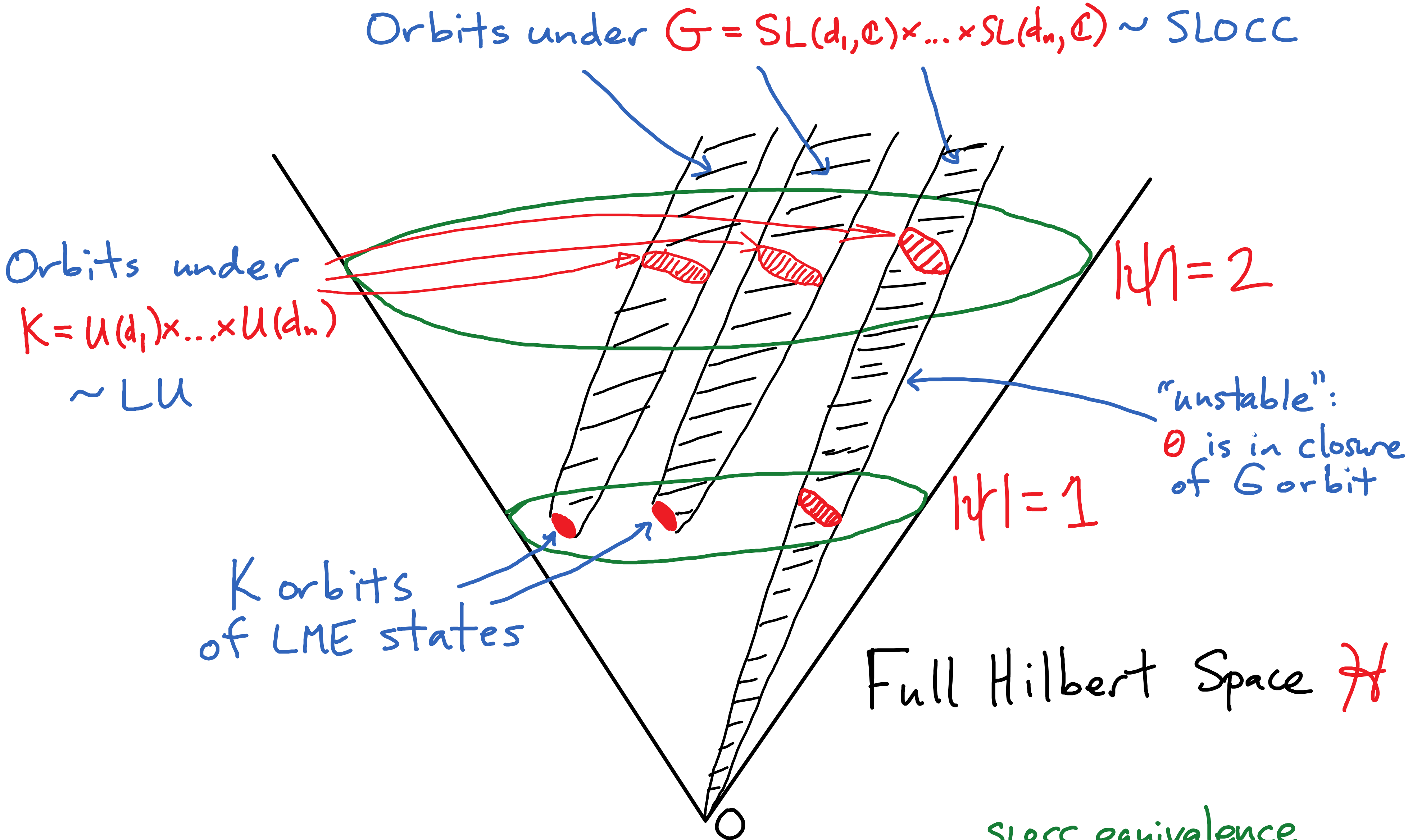
$$|\psi\rangle \to \mathcal{U}_1 \otimes \mathcal{U}_2 \otimes \dots \otimes \mathcal{U}_n |\psi\rangle \qquad \mathcal{U}_i \in U(d_i)$$

Natural to consider space $\mathcal{H}_{LME}/K$ of inequivalent LME states, where

$$K = \mathcal{U}(d_1) \times \dots \times \mathcal{U}(d_n)$$

Local Unitaries

⋆ This space is a Kähler manifold ⋆



$\mathcal{H}_{LME}/K$ is equivalent to $\mathcal{H}/\!/G$

(Kempf-Ness, Klyachko, ...)

SLOCC equivalence classes, excluding "unstable" states

$\parallel$

Geometric invariant theory quotient

Basic questions:

For which $(d_1, d_2, \ldots, d_N)$ do LME states exist?

How can we characterize the space $\mathcal{H}_{LME}/K$?
(dimension, geometry, etc...)

Can we give explicit constructions?

$\star$ related to natural questions in representation theory, symplectic geometry, and algebraic geometry/geometric invariant theory $\star$

Example: $n = 2$

- Schmidt decomposition: $|\psi\rangle = \sum_i \sqrt{p_i}\, \underset{\text{orthonormal}}{|\psi_i^1\rangle} \otimes \underset{\text{orthonormal}}{|\psi_i^2\rangle}$
- $\{p_i\}$ = nonzero eigenvalues of ρ_1 and ρ_2

$\therefore$ must have $d_1 = d_2 = d$, $p_i = \frac{1}{d}$

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_i |\psi_i^1\rangle \otimes |\psi_i^2\rangle$$

= Bell state. Unique up to local unitaries

Example: $n = 3$

Schmidt decomposition argument gives (from $\{p_i\}_{\rho_3} = \{p_i\}_{\rho_{12}}$)

$$d_1 \leq d_2 \cdot d_3, \quad d_2 \leq d_1 \cdot d_3, \quad d_3 \leq d_1 \cdot d_2$$

but these are NOT sufficient

Direct construction for dims. $2 \leq B \leq C$ shows LME states exist iff $(2, B, C) = (2, N, N)$ or $(2, NK, (N+1)K)$

e.g. $(2, 3, 5)$ impossible

A Representation Theory Construction

Let H be any group (finite or compact continuous) unitary irreducible representations $R_1, R_2, \ldots, R_n$ of dimension $d_1, d_2, \ldots, d_n$ such that

$$\underbrace{1}_{\text{trivial representation}} \subset R_1 \underbrace{\times \ldots \times}_{\text{tensor product of representations}} R_n$$

Then if $|0\rangle = \sum c_{i_1 \ldots i_n} |i_1\rangle \otimes \ldots \otimes |i_n\rangle$ gives the trivial representation, $|0\rangle$ gives a locally maximally entangled state in $\mathcal{H} = \mathcal{H}_{d_1} \otimes \ldots \otimes \mathcal{H}_{d_n}$.

PROOF:

Let $U_h \in U(d_i)$ be the representative of $h \in H$ acting on $\mathcal{H}_{d_i}$

Then $U_h \rho_i U_h^{-1} = \rho_i$ i.e. $[U_h, \rho_i] = 0$, for all $h \in H$.

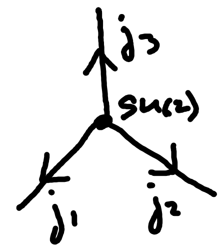
By Shur's lemma, ρ_i is a multiple of $\mathbb{1}$.

EXAMPLES:

① Take $H = SU(2)$ reps (j_1, j_2, j_3) with $j_3 \in j_1 \times j_2$

$$|0\rangle = \sum_{m_i} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |m_1\rangle \otimes |m_2\rangle \otimes |m_3\rangle$$

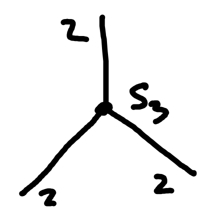
3j-symbol



gives LME state in $\mathcal{H}_{2j_1+1} \otimes \mathcal{H}_{2j_2+1} \otimes \mathcal{H}_{2j_3+1}$

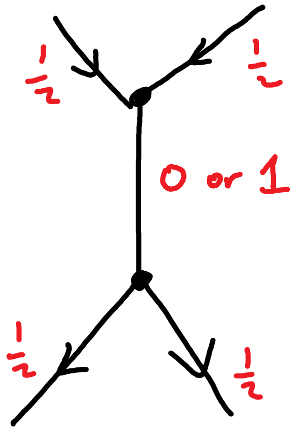
② Take $H = S_3$, $R_1 = R_2 = R_3 = \yng(2,1)$ (2 dimensional).

Then $|0\rangle = \sum_i C_{i_1 i_2 i_3} |i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle$ gives GHZ state

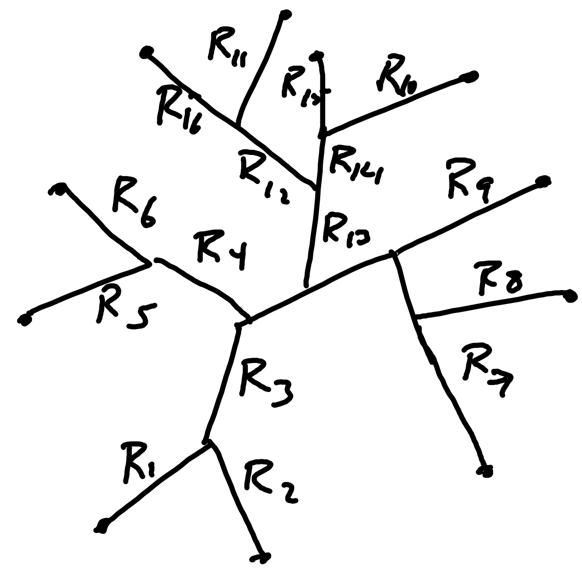


Examples, continued

④ $H = SU(2)$



⑤ general H

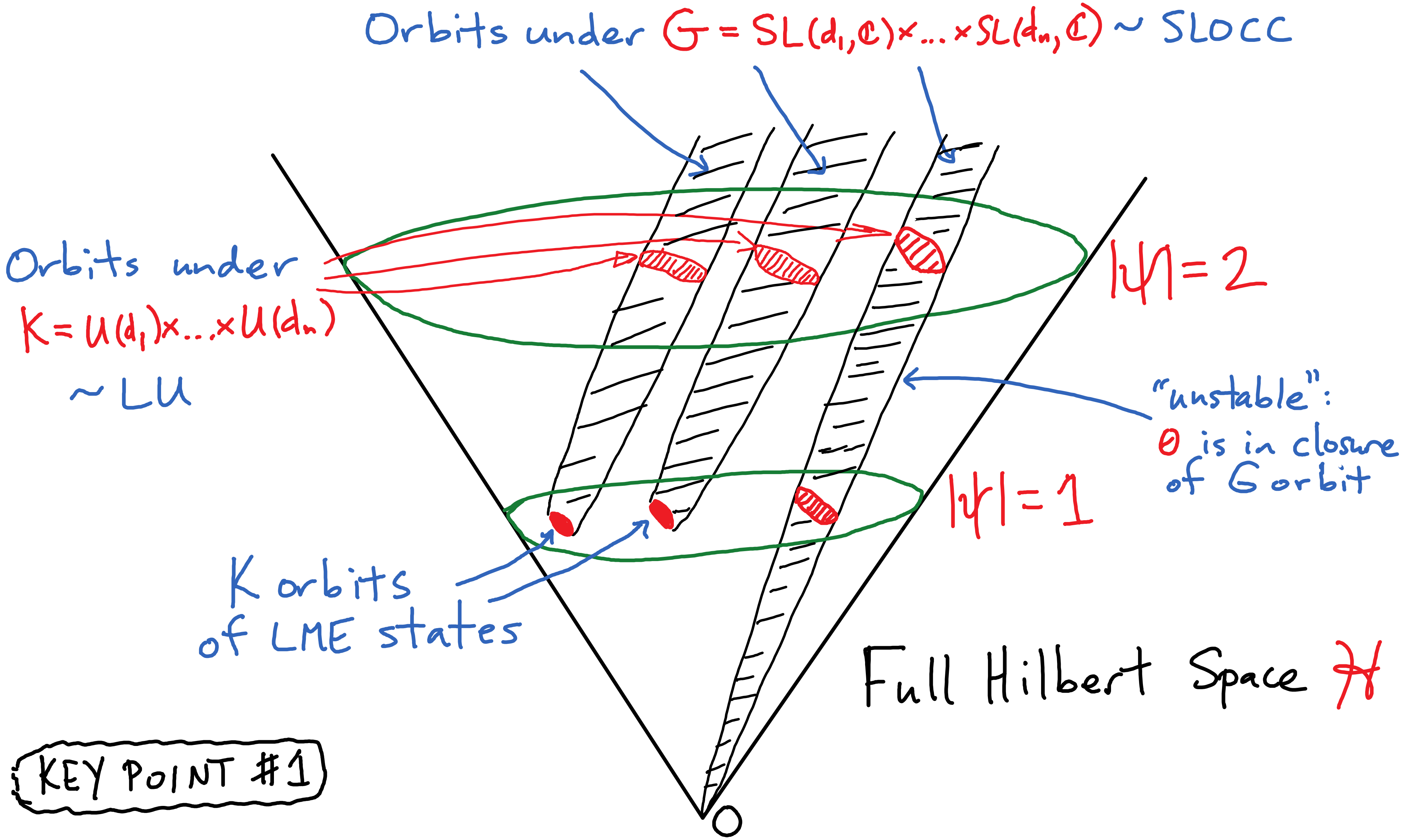


- Any way of constructing invariant from product of reps gives LME state $\longrightarrow$ can always represent by network with cubic vertices.

- Composite subsystems for which $\otimes_i R_i$ is irreducible are also maximally mixed $\rightarrow$ can get perfect tensors/absolutely max. ent. states

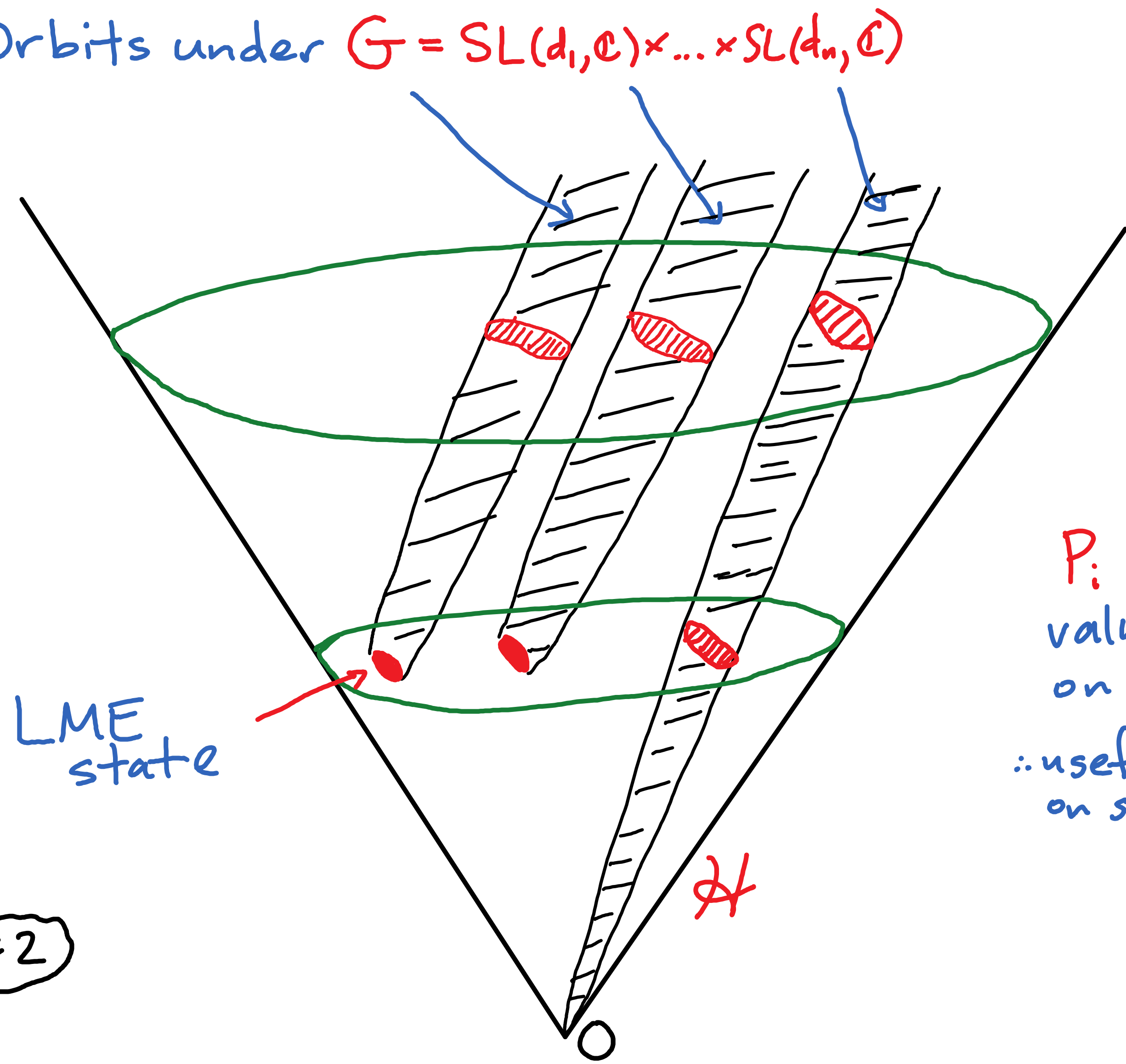
- If we gauge H acting irreducibly on all subsystems then ALL states are of this type. (Eliot Hijano)

CHARACTERIZING THE SPACE OF LME STATES



KEY POINT #1

$|\Psi\rangle \in \mathcal{H}_{LME}$ iff it is of minimum norm on its G-orbit $\rightsquigarrow$ $\mathcal{H}_{LME}/K \equiv \mathcal{H}/\!/G$



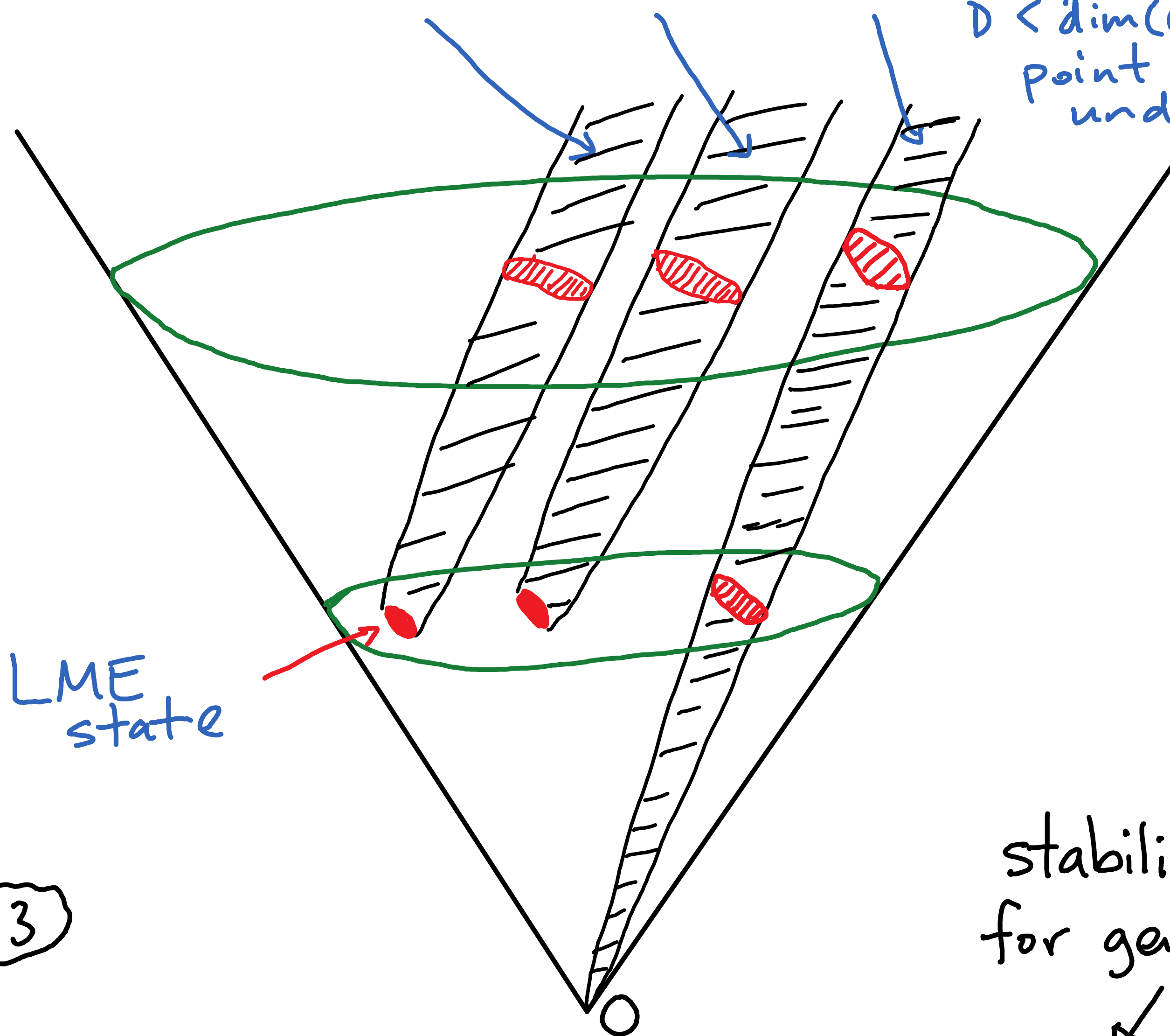
P_i has same value everywhere on a G-orbit
∴ useful coordinate on space of orbits.

KEY POINT #2

- $\mathcal{H}//G$ described algebraically by G-invariant polynomials $P_1, P_2, \ldots$ and their relations $f_n(P_i) = 0$

e.g.: $P = \det \psi_{ij}$ for $\mathcal{H}_d \times \mathcal{H}_d$

Orbits under $G = SL(d_1,\mathbb{C})\times\ldots\times SL(d_n,\mathbb{C})$: may have dimension $D < \dim(G)$ if generic point invariant under a subgroup



KEY POINT #3

stabilizer $S<G$ for generic point

$$\dim(\mathcal{H}_{LME}/K) = \dim(\mathcal{H}/\!/G) = \dim(\mathcal{H}) - \dim(G) + \dim(S)$$

- easy to compute in examples
- use general results by Elashvili/Popov

THEOREM (Bryan, Rechstein, V.R.)

For $\mathcal{H} = \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \dots \otimes \mathcal{H}_{d_n}$ with $d_1 \leqslant d_2 \leqslant \dots \leqslant d_n$:

d_{n-1} $\qquad$ $\frac{1}{2} d_1 \cdots d_{n-1}$ $\qquad$ $d_1 \cdots d_{n-1}$ $\qquad$ d_n

$\mathcal{H}_{LME}/K$ has naive dimension

$$\dim(\mathcal{H}) - \dim(G)$$
$$= d_1 \cdots d_n - d_1^2 - \dots - d_n^2 + n - 1$$

except:

$(2,2,2) \longrightarrow$ point

$(2,N,N) \longrightarrow \dim N-3$

from results on stabilizers

dimension same as for

$$\{d_1, \dots, d_{n-1}, d_1 \cdots d_{n-1} - d_n\}$$

from explicit map between invariant polynomials

$\mathcal{H}_{LME}/K$ is a point

$\mathcal{H}_{LME}$ is empty

gives recursive algorithm to find dimension.

THEOREM: (Bryan, Reichstein, VR)

all invariant under recursion step

Define invariants:

$$R = d_1 \cdots d_n - \sum_{k=1}^{n} (-1)^k \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \gcd^2(d_{i_1}, \ldots, d_{i_k})$$

$$\Delta = d_1 \cdots d_n - d_1^2 - \cdots - d_n^2 + n - 1$$
$$= \dim(\mathcal{H}) - \dim(G) - 1$$

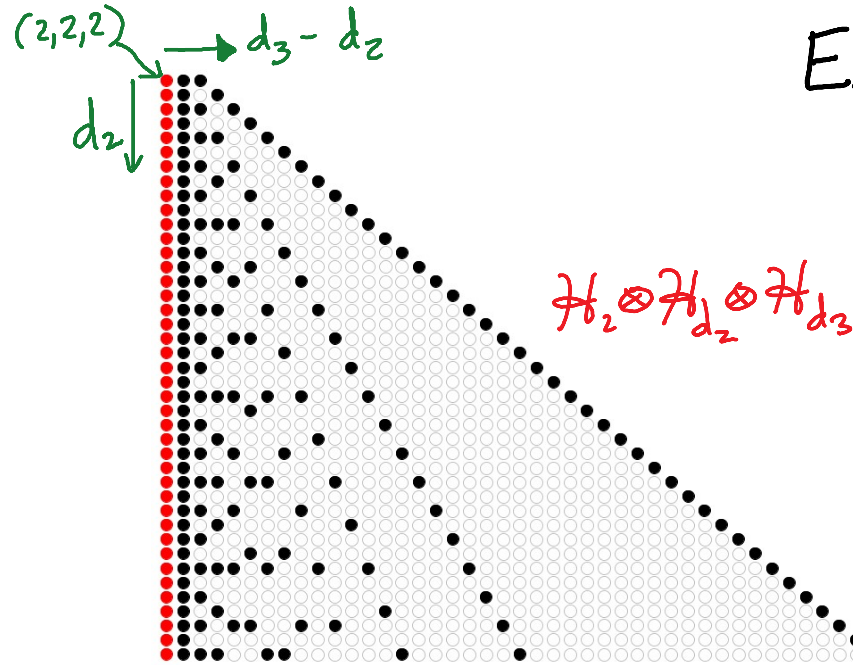
$$g_{max} = \max(\gcd(d_i, d_j))$$

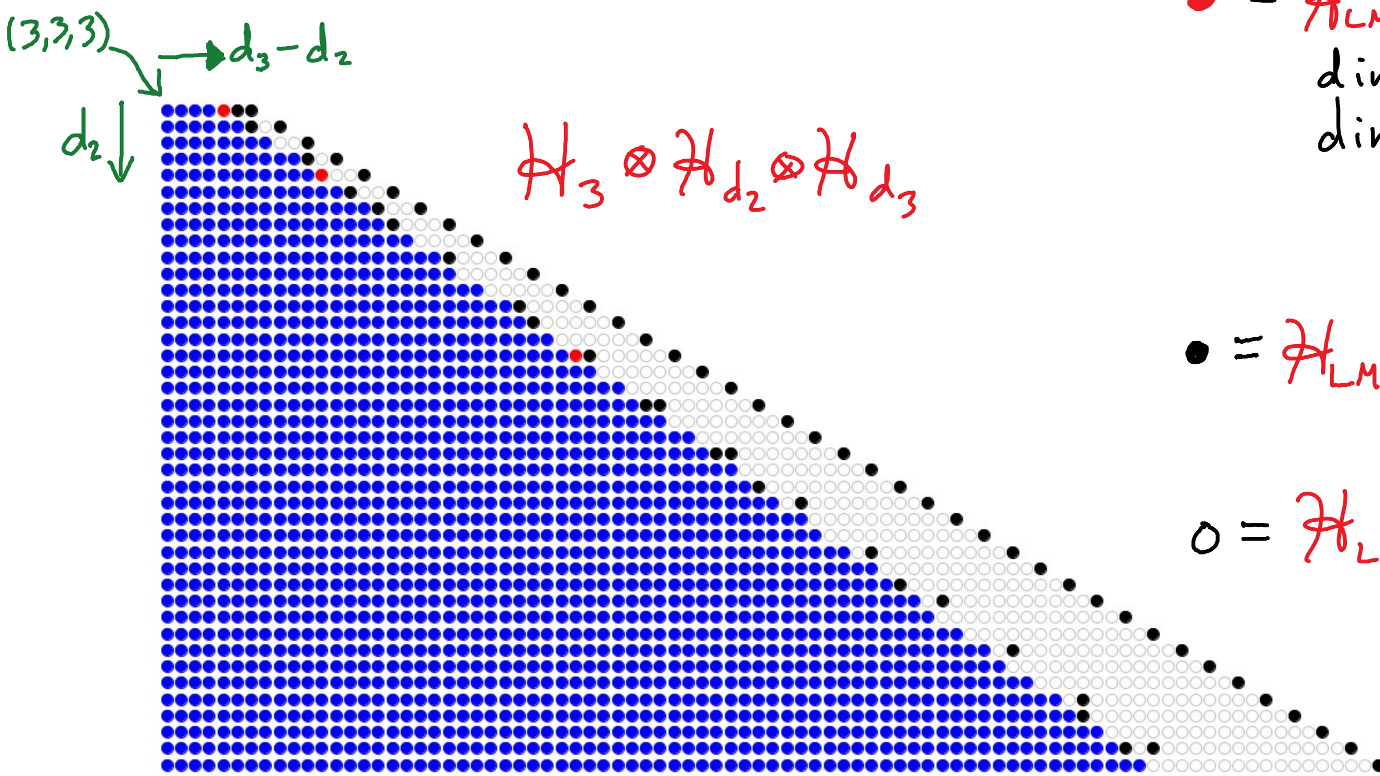
Then $\mathcal{H}_{LME}/K$ is non-empty if and only if $R \geq 0$

$R \geq 0$: Dimension is

Δ	for	$\Delta > -2$
$\max(g_{max} - 3, 0)$	for	$\Delta = -2$
0	for	$\Delta < -2$

Example Results for Tripartite Case





● (blue) = $\mathcal{H}_{LME}/K$ has expected dimension $\dim(\mathcal{H}) - \dim(G)$

● (red) = $\mathcal{H}_{LME}/K$ has expected dimension -2, actual dimension $\max(\gcd(d_i, d_j)) - 3$

● = $\mathcal{H}_{LME}/K$ is a single point

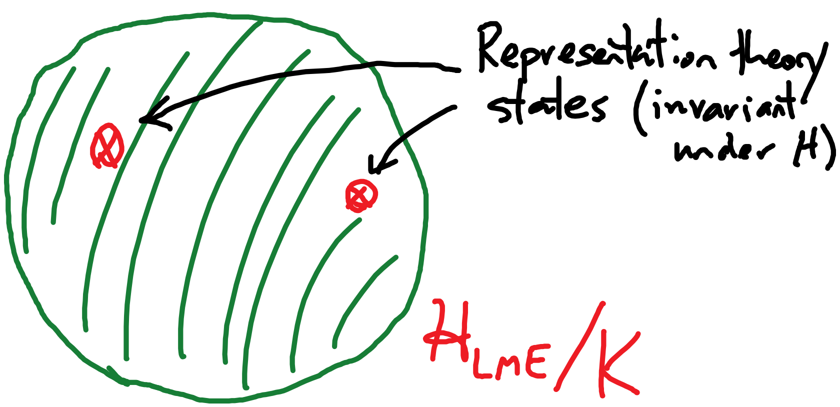
○ = $\mathcal{H}_{LME}/K$ is empty

Summary:

- Explicit construction of LME states via representation theory
  - these are special "stabilizer" states

- Calculated dimension of Kähler manifold $\mathcal{H}_{LME}/K \sim \mathcal{H}/\!\!/G$ in all cases $\Rightarrow$ necessary & sufficient conditions for existence

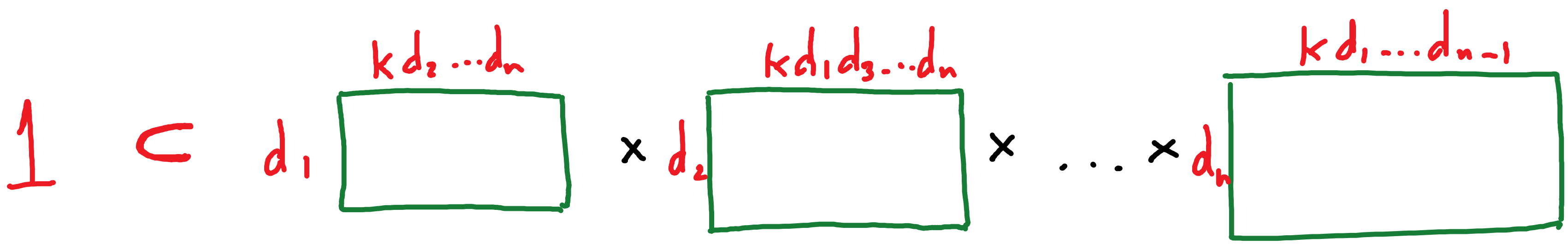
→ Also gives dimension of stabilizer in $SL(d_1,\mathbb{C}) \times \ldots \times SL(d_n,\mathbb{C})$ for generic state



ASIDE:

Quotient is non-empty if there exists a polynomial in $\psi_{i_1,\dots,i_n}$ invariant under $G = SL(d_1,\mathbb{C}) \times \dots \times SL(d_n,\mathbb{C})$

Equivalent (via Shur-Weyl duality): quotient is non-empty if for some k representations of $S_{kd_1\cdots d_n}$ obey



same condition from Klyachko solution of quantum marginal problem

But: computationally hard to decide this.

REPRESENTATION THEORY IMPLICATIONS:

There exists group H, unitary irreps $R_1, \dots, R_n$ of dimensions $d_1, \dots, d_n$ with $1 \subset R_1 \times \dots \times R_n$ only if $R(d_1, \dots, d_n) \geq 0$.

Also sufficient for $(2, A, B)$ case. Could $R \geq 0$ be necessary + sufficient in general?