Linear time algorithm(s) for Quantum 2SAT

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A joint talk on:

[ASSZ] Itai Arad, Miklos Santha, Aarthi Sundaram & Shengyu Zhang,
'Linear time algorithm for quantum 2SAT',
CoRR abs/1508.06340. (2015)

[dBG] Niel de Beaudrap & Sevag Gharibian,
'A linear time algorithm for quantum 2-SAT',
CoRR abs/1508.07338. (2015)

Background

Classical 2SAT

A classical 2SAT instance Φ is a Boolean formula defined on

- a set of n variables: $\{x_1, \ldots, x_n\}$
- as a conjunction of m clauses: $\{C_1, \ldots, C_n\}$ where
- each clause is an `OR` of at most 2 literals (i.e. x_i and $\overline{x}_i$)

Goal: Find an assignment to the variables so that Φ evaluates to `true`.

Example (2SAT)

An instance: $\Phi = (x_1 \vee x_2) \wedge (\overline{x}_1 \vee \overline{x}_2) \wedge (x_1 \vee \overline{x}_4) \wedge (x_4) \wedge (x_2 \vee x_3)$
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Algorithms for 2SAT

1. Even, Itai & Shamir (1976): A backtracking, resolution based search of possible assignments.
2. Apsvall, Plass & Tarjan (1979): Finds strongly connected components in the implication graph of the instance.

Both algorithms have an optimal $O(n + m)$ running time.

Quantum 2SAT (2-QSAT)

A quantum 2SAT instance $\mathcal{H}$ is a 2-local Hamiltonian defined on

- n qubits: $\{x_1, \ldots, x_n\}$
- as a sum of m local terms: $\mathcal{H} = \sum_{uv} \Pi_{uv}$ where
- each Π_{uv} is a projector acting non-trivially on qubits (u, v);

Example (Q2SAT)

A 2-QSAT *instance:* $\mathcal{H} = \Pi_{12} + \Pi_{23} + \Pi_{34}$ *with*

$\Pi_{12} = |00\rangle\langle 00| + |11\rangle\langle 11|; \qquad \Pi_{34} = |\Psi^-\rangle\langle\Psi^-|; \qquad \Pi_{23} = |01\rangle\langle 01|;$

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- each Π_{uv} is a projector acting non-trivially on qubits (u, v);
- the smallest eigenvalue of $\mathcal{H}$ is its ground energy and
- the corresponding eigenvector $|\psi\rangle$ is the ground state of $\mathcal{H}$.

Goal: Given $\mathcal{H}$, output a ground state if the ground energy is 0 or "Unsatisfiable" otherwise.

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Solving 2-QSAT

Prior Work

An $O(n^4)$ 2-QSAT algorithm by Bravyi (2006) based on finding the transitive closure of a directed graph.

- For $k \geq 3$, k-QSAT is QMA_1-complete

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- Quantum variant of the EIS Algorithm [ASSZ]
  - Infers a qubit assignment and propagates it throughout the system.
- Quantum analogue of the APT algorithm [dBG]
  - Uses the notion of transfer matrices to mirror the implications in Boolean Formulae.

Improvements in run-time

Bravyi's algorithm uses the following approach:

- For every qubit triple (i, j, k), if there is a constraint on (i, j) and (j, k), add an implied constraint acting on (i, k).
- Takes $O(n^3)$ time to add all implied constraints
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The linear time algorithms approach an instance by: (from [ASSZ, dBG])

- Analyzing only a part of the instance and manipulating it with local operations.
- Local sections of the instance on being solved are decoupled from the rest of instance.
- Governed by graph traversals that can be executed in linear time.

Algorithm Building Blocks

Preliminaries

Assume WLOG that $\mathcal{H}$ contains rank-1 and rank-3 projectors only.

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- n vertices representing each qubit and
- labeled edges Π_{ij} between (i,j) for each term in $\mathcal{H}$

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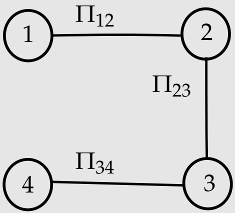

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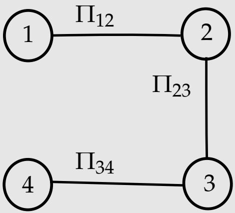

Theorem (Product State Theorem [CCD+11,ASSZ15])

Any satisfiable 2-QSAT *instance has a ground state which is a tensor product of one qubit and two-qubit states, where two-qubit states only appear in the support of rank-3 projectors.*

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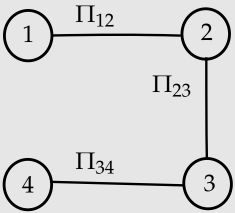
Ground State $= |\Psi^+\rangle_{12} \otimes |0\rangle_3 \otimes |1\rangle_4$



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Definition (Propagation)

Let $\Pi_{ij} = |\psi\rangle\langle\psi|$ *be a rank-*1 *projector and* $|\alpha\rangle$ *be the state assigned to* i. *Then,* Π_{ij} *propagates* $|\alpha\rangle$ *if, up to a phase, there exists a unique* 1*-qubit state* $|\beta\rangle$ *such that* $\langle\psi|(|\alpha\rangle_i \otimes |\beta\rangle_j) = 0$.

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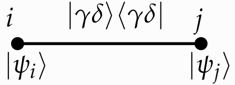

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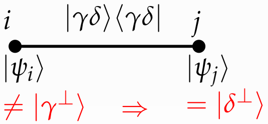

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i — $|\gamma\delta\rangle\langle\gamma\delta|$ — j

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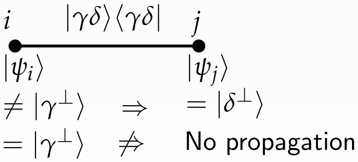

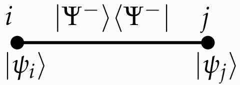

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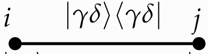

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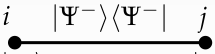

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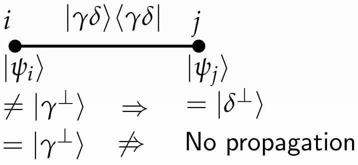

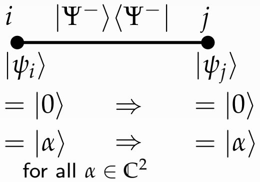

A product constraint will not propagate a state when already satisfied.

An entangled constraint always propagates every state.

Multi-qubit Propagation

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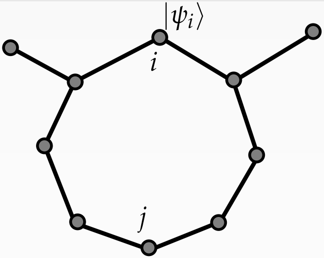

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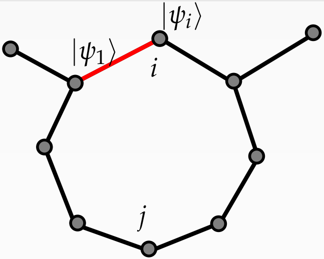

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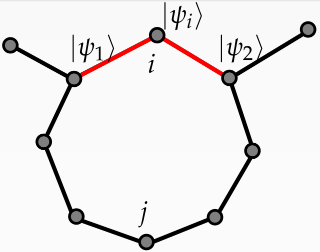

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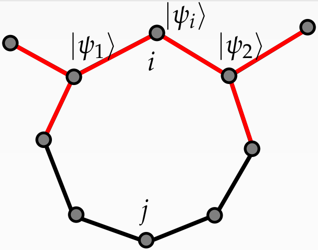

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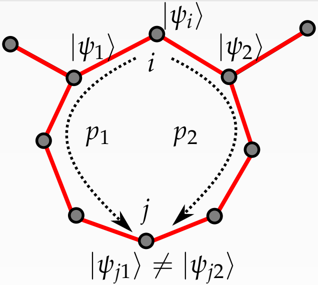

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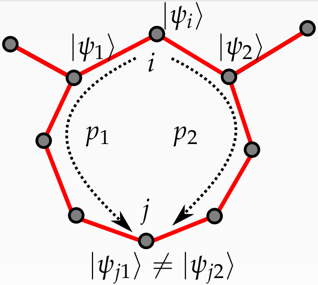

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Lemma (Propagation Lemma, Informal Statement)

Let the propagation $(i, |\psi_i\rangle)$ on $G(\mathcal{H})$ extend the assignment to $|\psi_i\rangle \otimes |\Phi\rangle$. If the propagation is:

1. *"Unsuccessful", there is no solution of the form $|\psi_i\rangle \otimes |rest\rangle$.*
2. *Otherwise, there exists a solution of the form $|\psi_i\rangle \otimes |\Phi\rangle \otimes |rest\rangle$.*

Algorithm Sketch

Part A: Rank-3 and Rank-1 Product Constraints

2-QSAT`Solver`$(G(\mathcal{H}))$

Step 1 For all rank-3 constraints Π_{ij} in $(G(\mathcal{H}))$

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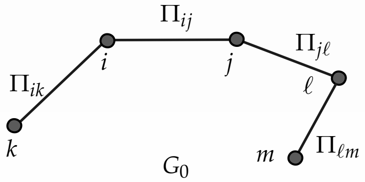

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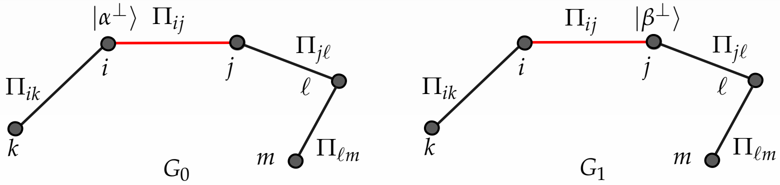

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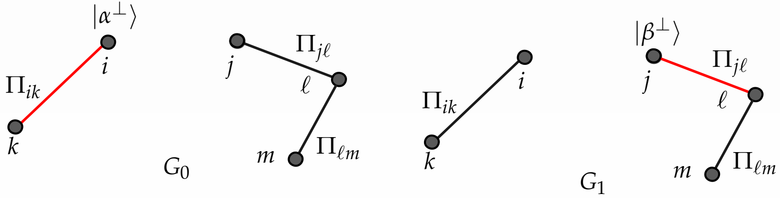

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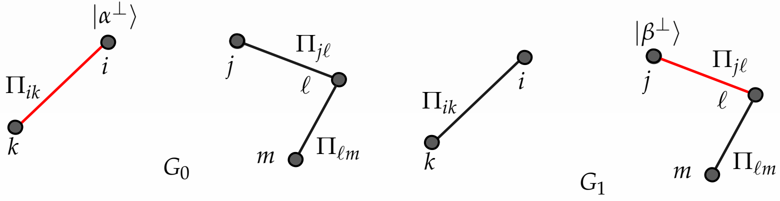

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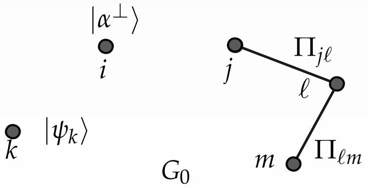

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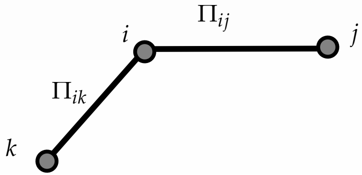

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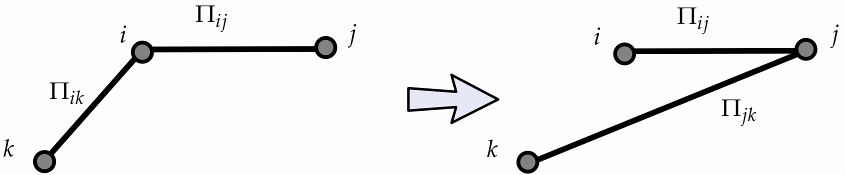

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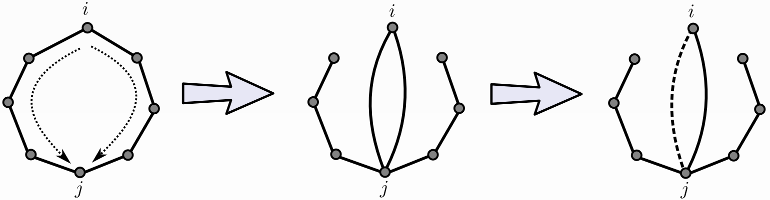

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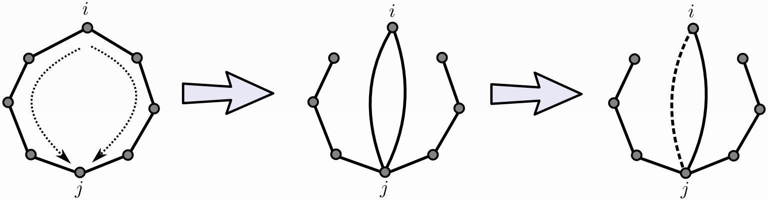

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Any 2-dimensional subspace V of the 2-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$ contains at least one product state, which can be found in constant time.

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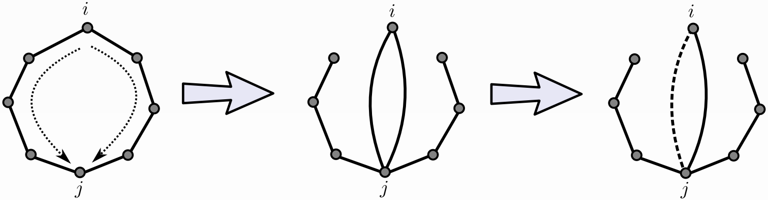

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- To deal with cycles of entangled constraints, each edge in the paths p_1, p_2 is considered at most 4 times – first propagation + sliding + parallel propagation.
- The sliding is constrained by the length of the cycle as each slide can be done in constant time.

Alternate method to deal with entangled constraints

Alternate Method from [dBG]

Step 5' Perform a depth first search on each remaining connected component.

Step 6' Find a discretizing cycle $C = (i, v_1, \ldots, j, \ldots, v_\ell, i)$ if it exists

i.e. transfer matrix $T_C = T_{v_\ell i} \ldots T_{v_1 v_2} T_{i v_1}$ has a non-degenerate spectrum

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Note that this method does not modify the constraint graph.

Cost of performing operations on complex numbers

Bit Complexity

Bit complexity refers to the number of bit-wise operations performed in the course of an algorithm.

Frequently used to analyze algorithms that manipulate algebraic and complex numbers.

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Main Result (from [dBG])

- Bit complexity of the algorithms is $O((m+n)M(n))$
  where $M(n) :=$ Cost of multiplying two n bit numbers.
- Explicit constructions show that $O(n+m)$ bit complexity is not possible for general 2-QSAT instances.
- When all constraints are product, the bit complexity matches the linear bit complexity of 2SAT

Thanks for your attention!