Quantum linear systems algorithm with exponentially improved dependence on precision

Andrew Childs
(U. Maryland)

Robin Kothari
(MIT)

Rolando Somma
(Los Alamos)

arXiv:1511.02306
QIP 2016

Summary

Known [HHL09]: Quantum algorithm to solve* (to error ϵ) a system of $N \times N$ linear equations*

Running time is $\mathrm{poly}(\log N, 1/\epsilon)$

Our result: Running time can be improved to $\mathrm{poly}(\log N, \log(1/\epsilon))$ for the same problem

Solving linear equations

Input: Hermitian matrix A in $\mathbb{C}^{N\times N}$ and vector $\vec{b}$ in $\mathbb{C}^{N}$

Goal: To solve the equation

$$A\vec{x} = \vec{b}$$

i.e., to compute (approximately) $\vec{x} = A^{-1}\vec{b}$

How are the inputs and outputs represented?

If A, $\vec{b}$, and $\vec{x}$ are written explicitly, then

- Classically, we can solve this in time poly(N)
- Quantumly, we need poly(N) time ⇒ no exponential speedup

Quantum linear systems problem (QLSP)

Goal: To solve the equation (given Hermitian A in $\mathbb{C}^{N\times N}$ and $\vec{b}$ in $\mathbb{C}^{N}$)

$$A\vec{x} = \vec{b}$$

i.e., to compute (approximately) $\vec{x} = A^{-1}\vec{b}$

Modified problem

1. Assume A is
   - Sparse: At most polylog N nonzero entries per row/column
   - Row computable: Nonzero entries computable in time polylog N
   - Well conditioned: Condition number $\kappa \coloneqq \lambda_{\max}/\lambda_{\min}$ is polylog N
2. Assume $|b\rangle \coloneqq \vec{b}/\|\vec{b}\|$ can be created efficiently (time polylog N)
3. New Goal: Create the quantum state $|x\rangle \coloneqq \vec{x}/\|\vec{x}\|$ up to error ϵ

Known results

Theorem [HarrowHassidimLloyd09]
Let A be d-sparse, $\|A\| = 1$, and have condition number κ.
QLSP can be solved in time $O(d\kappa^2/\epsilon \text{ polylog}(Nd\kappa/\epsilon))$

Tools: Hamiltonian simulation, phase estimation (source of $1/\epsilon$)

Theorem [Ambainis12]
Improved dependence on κ: $O(d\kappa/\epsilon^3 \text{ polylog}(Nd\kappa/\epsilon))$

Tools: Variable-time amplitude amplification (uses phase est.)

Applications: Solving differential equations [Berry10]
Machine learning [WiebeBraunLloyd12, LloydMohseniRebentrost13]
Computing scattering cross sections [CladerJacobsSprouse13]
Computing effective resistance [Wang13]

Our results

Theorem [HarrowHassidimLloyd09]
Let A be d-sparse, $\|A\| = 1$, and have condition number κ.
QLSP can be solved in time $O(d\kappa^2/\cancel{\epsilon}\ \mathrm{polylog}(Nd\kappa/\epsilon))$

Our Result: Running time $O(d\kappa^2 \quad \mathrm{polylog}(Nd\kappa/\epsilon))$

Theorem [Ambainis12]
Improved dependence on κ: $O(d\kappa/\cancel{\epsilon}^3\ \mathrm{polylog}(Nd\kappa/\epsilon))$

Our Result: Running time $O(d\kappa \quad \mathrm{polylog}(Nd\kappa/\epsilon))$

Why $\log(1/\epsilon)$?

- Useful when using QLSP as a subroutine
- Natural scaling for many problems of interest
- Complexity-theoretic implications (e.g., [FeffermanLin16])

Part 2: techniques

Linear combination of unitaries

$V = \alpha_0 U_0 + \alpha_1 U_1 + \cdots$ Let $\|\alpha\|_1 := \sum_i |\alpha_i|$

Map we want to perform — Easy-to-perform unitaries

Goal: To create the state $V|b\rangle / \|V|b\rangle\|$.

Linear combination of unitaries Lemma

Let $V = \sum_i \alpha_i U_i$, c-U $= \sum_i |i\rangle\langle i| \otimes U_i$, and Ref$(|b\rangle) = I - 2|b\rangle\langle b|$.

We can create $\frac{V|b\rangle}{\|V|b\rangle\|}$ with $O\left(\frac{\|\alpha\|_1}{\|V|b\rangle\|}\right)$ uses of c-U and Ref$(|b\rangle)$.

Strategy: Express $V = A^{-1}$ as a linear combination of easy-to-perform unitaries.

Linear combination of unitaries

Corollary: If $A^{-1} = \sum_i \alpha_i U_i$, we can prepare $|x\rangle = \frac{A^{-1}|b\rangle}{\| A^{-1}|b\rangle \|}$ with $O(\|\alpha\|_1)$ uses of c-U and Ref($|b\rangle$).

Choices for U_i:

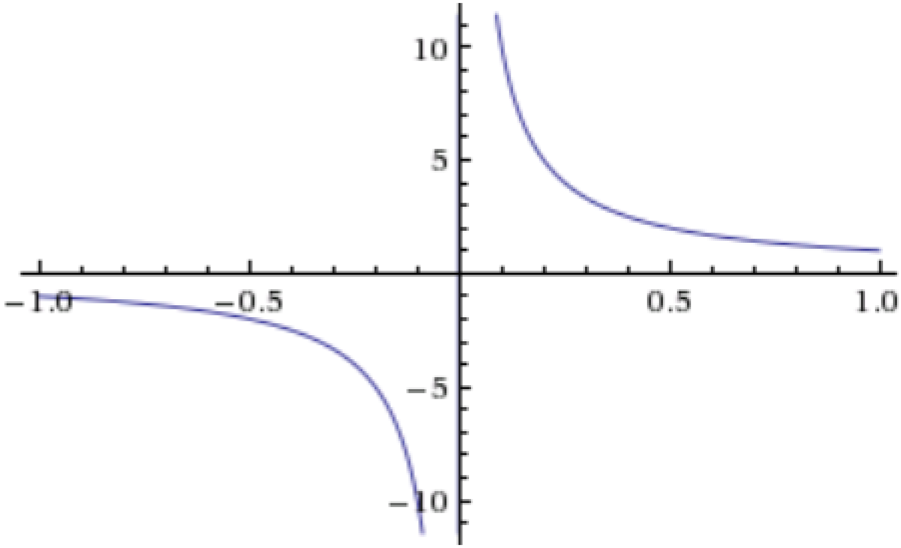
1. Use $\exp(-iAt)$: $A^{-1} = \sum_t \alpha_t \exp(-iAt)$
   Implement $\exp(-iAt)$ using Hamiltonian simulation
   We call this the Fourier approach

2. Use $T_n\left(\frac{A}{d}\right)$, where $T_n(\cdot)$ is the n^th Chebyshev polynomial
   Implement using quantum walk [Szegedy04, Childs10]
   We call this the Chebyshev approach

Fourier approach

Goal: Express $A^{-1} = \sum_t \alpha_t \exp(-iAt)$

Equivalent to expressing $x^{-1} = \sum_t \alpha_t \exp(-ixt)$



Function blows up at origin ⇒ no Fourier series expansion

Only needs to work in domain $D_\kappa = \left[-1, -\frac{1}{\kappa}\right] \cup \left[+\frac{1}{\kappa}, +1\right]$.

Fourier approach

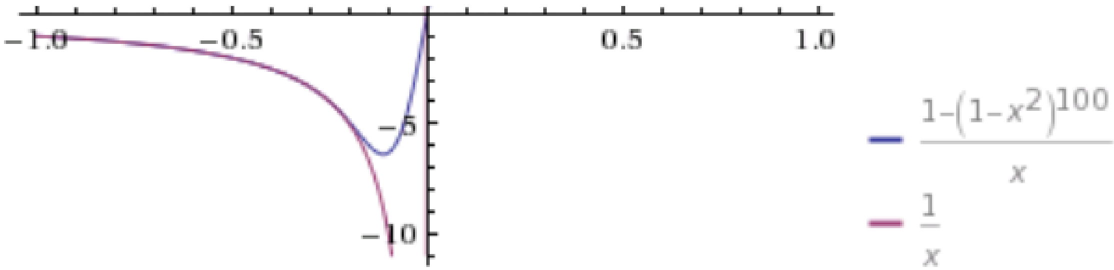
Goal: Express $x^{-1} = \sum_t \alpha_t \exp(-ixt)$ on $D_\kappa = \left[-1, -\frac{1}{\kappa}\right] \cup \left[\frac{1}{\kappa}, 1\right]$

Lemma 11. *Let the function $h(x)$ be defined as*

$$h(x) := \frac{i}{\sqrt{2\pi}} \sum_{j=0}^{J-1} \delta_y \sum_{k=-K}^{K} \delta_z \, z_k e^{-z_k^2/2} e^{-ixy_j z_k},$$

where $y_j := j\delta_y$, $z_k := k\delta_z$, for some $J = \Theta(\frac{\kappa}{\epsilon}\log(\kappa/\epsilon))$, $K = \Theta(\kappa \log(\kappa/\epsilon))$, $\delta_y = \Theta(\epsilon/\sqrt{\log(\kappa/\epsilon)})$ and $\delta_z = \Theta((\kappa\sqrt{\log(\kappa/\epsilon)})^{-1})$.

Then $h(x)$ is is ϵ-close to $1/x$ on the domain D_κ.



Chebyshev approach

Goal: Express $A^{-1} = \sum_n \alpha_n T_n\left(\frac{A}{d}\right)$.

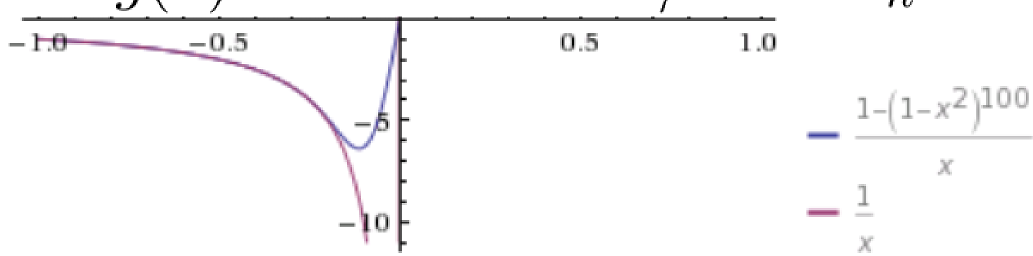
Equivalent to expressing $x^{-1} = \sum_n \alpha_n T_n\left(\frac{x}{d}\right)$.

$1-(1-x^2)^{100}$

Lemma 14. *Let $g(x)$ be defined as*

$$g(x) := 4\sum_{j=0}^{j_0}(-1)^j\left[\frac{\sum_{i=j+1}^{b}\binom{2b}{b+i}}{2^{2b}}\right]\mathcal{T}_{2j+1}(x),$$

where $j_0 = 2\sqrt{b\log(4b/\epsilon)}$ *and* $b = \kappa^2\log(\kappa/\epsilon)$. *Then $g(x)$ is 2ϵ-close to $1/x$ on D_κ.*



Summary

Known [HHL09]: Quantum algorithm to solve* (to error ϵ) a system of $N \times N$ linear equations*

Running time is $\text{poly}(\log N, 1/\epsilon)$

Our result: Running time can be improved to $\text{poly}(\log N, \log(1/\epsilon))$ for the same problem

Comparison of approaches

Fourier approach

1. Uses Hamiltonian simulation as a black box, hence more general
2. Less efficient
3. Easier to understand?

Chebyshev approach

1. Uses a discrete-time quantum walk, which requires matrix entries of A, hence less general
2. More efficient
3. Easier to understand?

Open problems

- Find applications of QLSP!!

- What other functions of a matrix A can we implement?
  - Chebyshev polynomial of A
  - $\exp(iAt)$
  - ???

Discrete-time quantum walk for A

Let A be d-sparse with largest entry 1.

$$|\psi_j\rangle := |j\rangle \otimes \frac{1}{\sqrt{d}} \sum_{k\in[N]} \left(\sqrt{A^*_{jk}}|k\rangle + \sqrt{1-|A_{jk}|}\,|k+N\rangle \right)$$

$$T := \sum_{j\in[N]} |\psi_j\rangle\langle j|$$

$$W := S(2TT^\dagger - \mathbb{1})$$

$$W^n = \begin{pmatrix} \mathcal{T}_n(A/d) & -\sqrt{\mathbb{1}-(A/d)^2}\,\mathcal{U}_{n-1}(A/d) \\ \sqrt{\mathbb{1}-(A/d)^2}\,\mathcal{U}_{n-1}(A/d) & \mathcal{T}_n(A/d) \end{pmatrix}$$