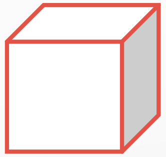
THE COMPLEXITY OF TRANSLATIONALLY INVARIANT SPIN CHAINS WITH LOW LOCAL DIMENSION

JOHANNES BAUSCH

TOBY CUBITT

MARIS OZOLS

HAMILTONIAN COMPLEXITY



physical system,
described by Hamiltonian $\mathbf{H}$

does $\mathbf{H}$ have lowest eigenvalue
$\lambda_{\text{min}} \leq \alpha$ or $\lambda_{\text{min}} \geq \beta$

scaling of promise gap $\beta - \alpha$

$\mathbb{C}^2$	*Kitaev '99*	5-local, arbitrary graph	QMA
$\mathbb{C}^2$	*Kempe, Kitaev, Regev '06*	2-local, arbitrary	QMA
$\mathbb{C}^2$	*Oliveira, Terhal '08*	2-local, 2D planar	QMA
$\mathbb{C}^{12}$	*Aharonov, Gottesman, Irani, Kempe '09*	2-local, line	QMA
$\mathbb{C}^{\text{huge}}$	*Gottesman, Irani '09*	2-local, line, translationally invariant	QMA_{EXP}
$\mathbb{C}^{41}$	*Bausch, Cubitt, Ozols '16*	2-local, line, translationally invariant	QMA_{EXP}

HAMILTONIAN COMPLEXITY

	VERIFIER RUNTIME	HAMILTONIAN GAP	
QMA	**poly** $\ln N$	$\dfrac{1}{\text{poly} \ln N}$	UNDERSPECIFIED
QMA_{EXP}	**exp poly** $\ln N \equiv \text{poly} N$	$\dfrac{1}{\text{poly} N}$	CORRECT CLASS

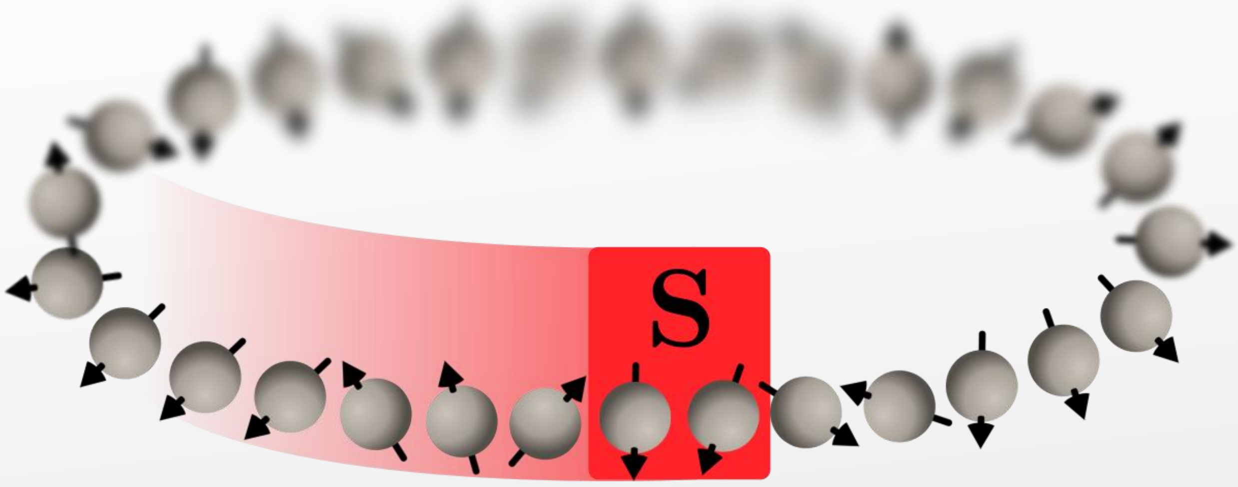
spin chain of length N
parameter N can encode
input of size $\ln N$

we want this!

OVERVIEW

COMPUTATIONAL MODEL	CONSTRUCTION	HAMILTONIAN GROUND STATE
Quantum Ring Machine ⇐	Quantum Thue System	Unitary Label Graph

QUANTUM RING MACHINES



Quantum Ring Machine. $(\mathbf{S}, n)$, $\mathbf{S}$ unitary operator on $(\mathbb{C}^d)^{\otimes 2}$, $n \in \mathbb{N}$.

$\mathbf{S}$ acts cyclicly on two neighbouring spins of the 1D spin chain $(\mathbb{C}^d)^{\otimes n}$.

Start computation in some initial configuration q_{i}.
Computation terminates in some final configuration q_{f}.

Runtime defined as for Turing Machines

QUANTUM RING MACHINES

Proof: embed a BQEXP-complete Turing Machine

Theorem. Let L be a promise problem in BQEXP.
Then there exists a polynomial p and a unitary $\mathbf{U}$ such that for each $x \in L$, the quantum ring machine $(\mathbf{U}, p(\exp|x|))$ terminates in $p(\exp|x|)$ steps.
On input $x \in L_{\text{YES}}$, it transitions to an accepting state with probability $\geq 2/3$, and analogously for NO instances.

OVERVIEW

COMPUTATIONAL MODEL

Quantum
Ring Machine

CONSTRUCTION

Quantum
Thue System

HAMILTONIAN GROUND STATE

Unitary
Label Graph

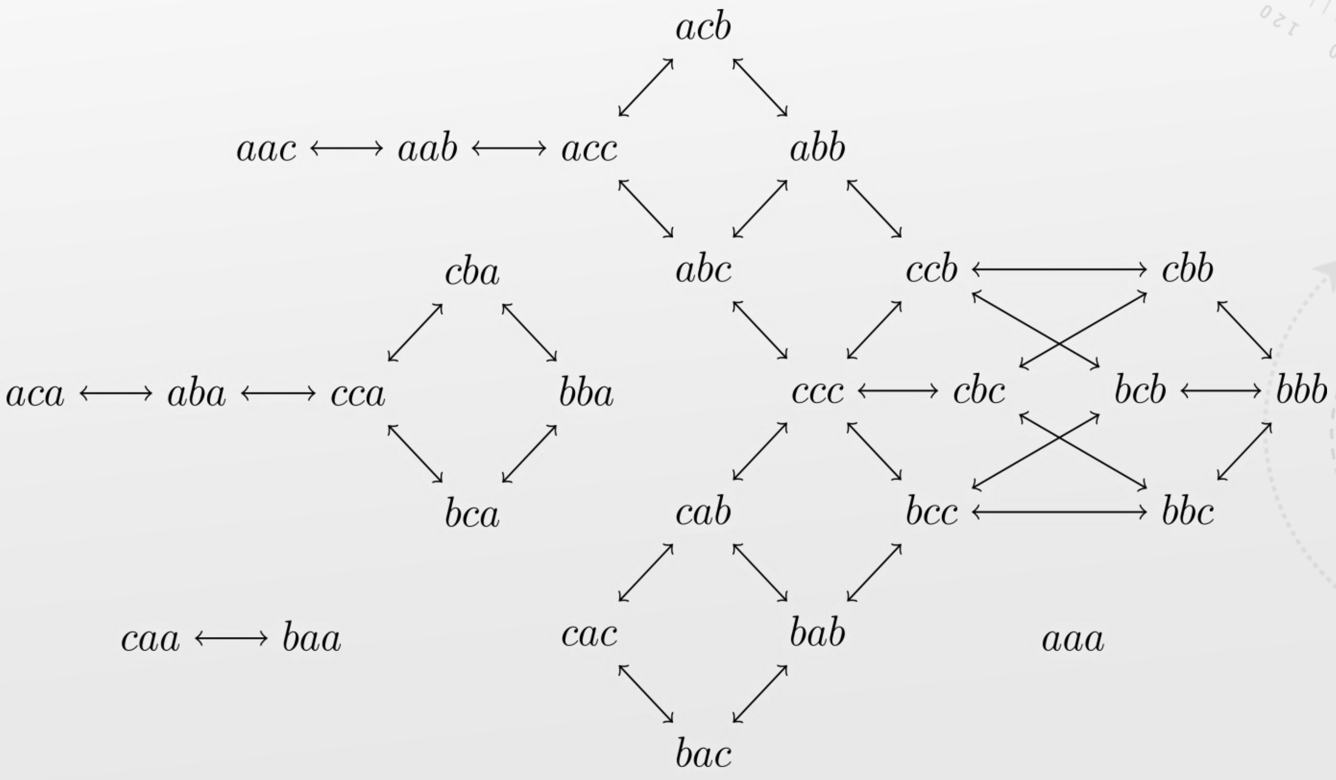
QUANTUM THUE SYSTEMS

[tuː]

finite index set

Thue System.
Finite alphabet Σ, set of length-preserving rules $\{(a_i \leftrightarrow b_i), i \in I\}$ with $a_i, b_i \in \Sigma^*$.

Example.
alphabet $\Sigma = \{a, b, c\}$
rules $\{(c \leftrightarrow b), (ab \leftrightarrow cc)\}$.



QUANTUM THUE SYSTEMS

classical

quantum

Quantum Thue System.
Thue system (Σ, R) with $\Sigma = \Sigma_{\mathsf{cl}} \dot{\cup} \Sigma_{\mathsf{q}}$ invariant under rules R, Hilbert space $\mathcal{H}$, family of unitaries for each rule $\{\mathbf{U}_r\}_{r\in R}$ such that $\mathbf{U}_r \in \mathcal{B}(\mathcal{H}^{\otimes|r|_{\mathsf{q}}})$.

number of quantum symbols in rule

Example. $\Sigma = \{-, *, |\}$, $\Sigma_{\mathsf{q}} = \{*\}$, $\mathcal{H} = \mathbb{C}^2$, $R = \{(*- \leftrightarrow -*, \sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|)\}$

$$* \underbrace{- \ldots -}_{n \text{ times}} | \quad \mapsto \quad - * \ldots - | \quad \mapsto \quad \cdots \quad \mapsto \quad - \ldots - *|$$

Starting on $|1\rangle$, end up in $\begin{cases} |1\rangle & \text{if } n \text{ is even} \\ |0\rangle & \text{otherwise.} \end{cases}$ $\rightarrow$ Decides whether n is even or odd.

OVERVIEW

COMPUTATIONAL MODEL

Quantum Ring Machine

implements a BQEXP complete

Turing's Wheelbarrow

CONSTRUCTION

Quantum Thue System

is a 2-local

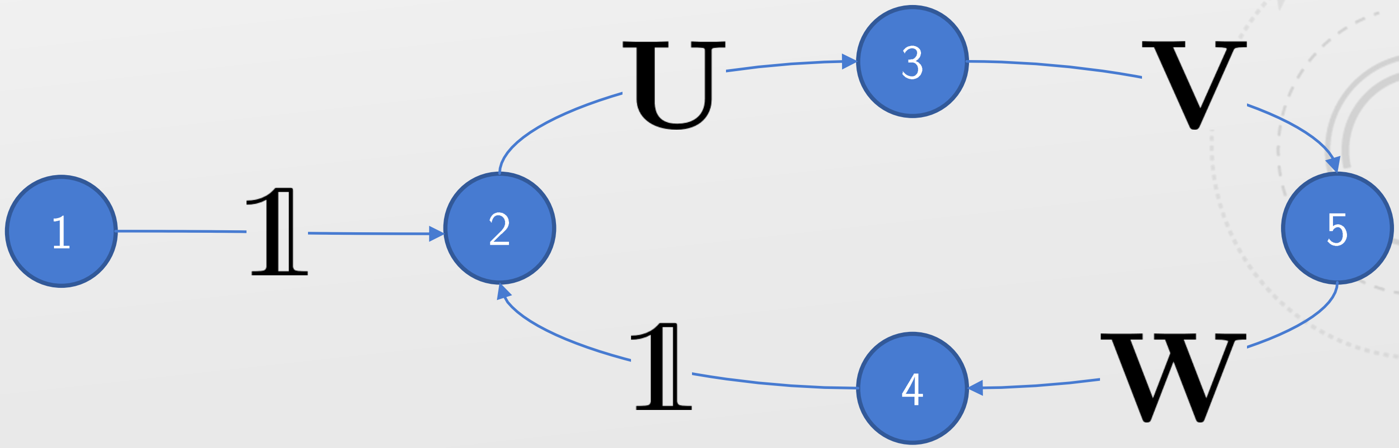
HAMILTONIAN GROUND STATE

Unitary Label Graph

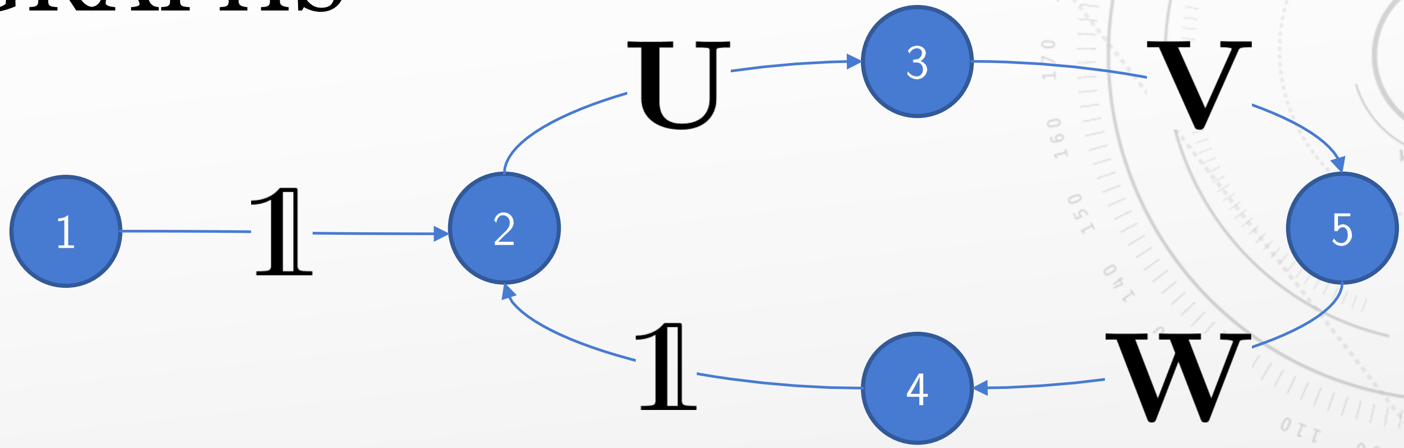
UNITARY LABEL GRAPHS

Definition (ULG).

- a directed graph $G = (V, E)$
- a family of Hilbert spaces $(\mathcal{H}_v)_{v \in V}$
- a family of unitary operators $(\mathbf{U}_e : \mathcal{H}_a \longrightarrow \mathcal{H}_b)_{e=(a,b) \in E}$



UNITARY LABEL GRAPHS



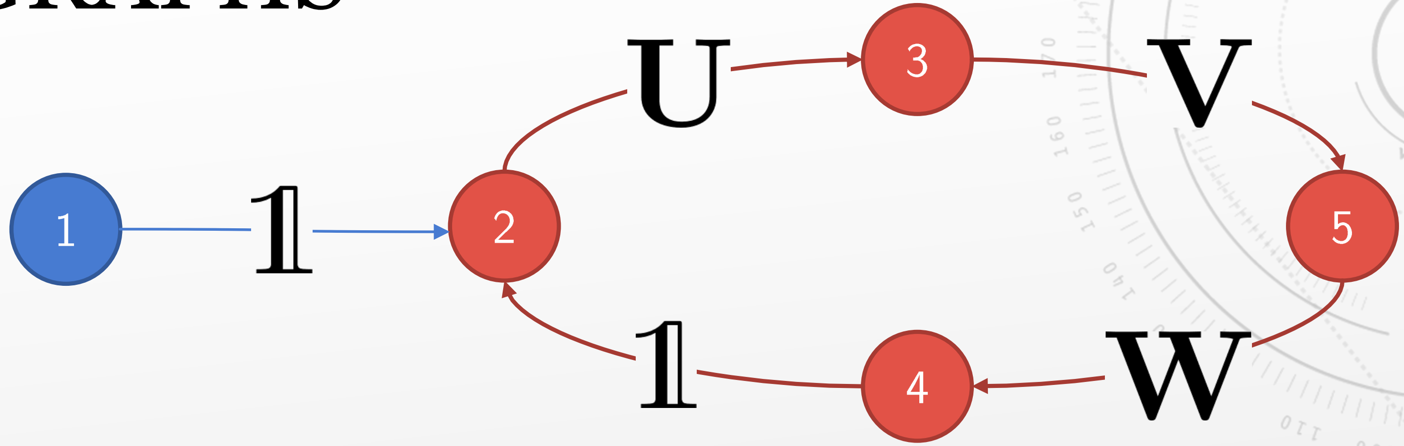
Graph Laplacian

$$\Delta = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix} = \sum_{(a,b)\in E} (|a\rangle - |b\rangle)(\langle a| - \langle b|)$$

$$\mathbf{H} = \sum_{e=(a,b)\in E} \sum_i (|a\rangle \otimes |i\rangle - |b\rangle \otimes \mathbf{U}_e |i\rangle)(\quad \text{h.c.} \quad)$$

ULG Hamiltonian

UNITARY LABEL GRAPHS

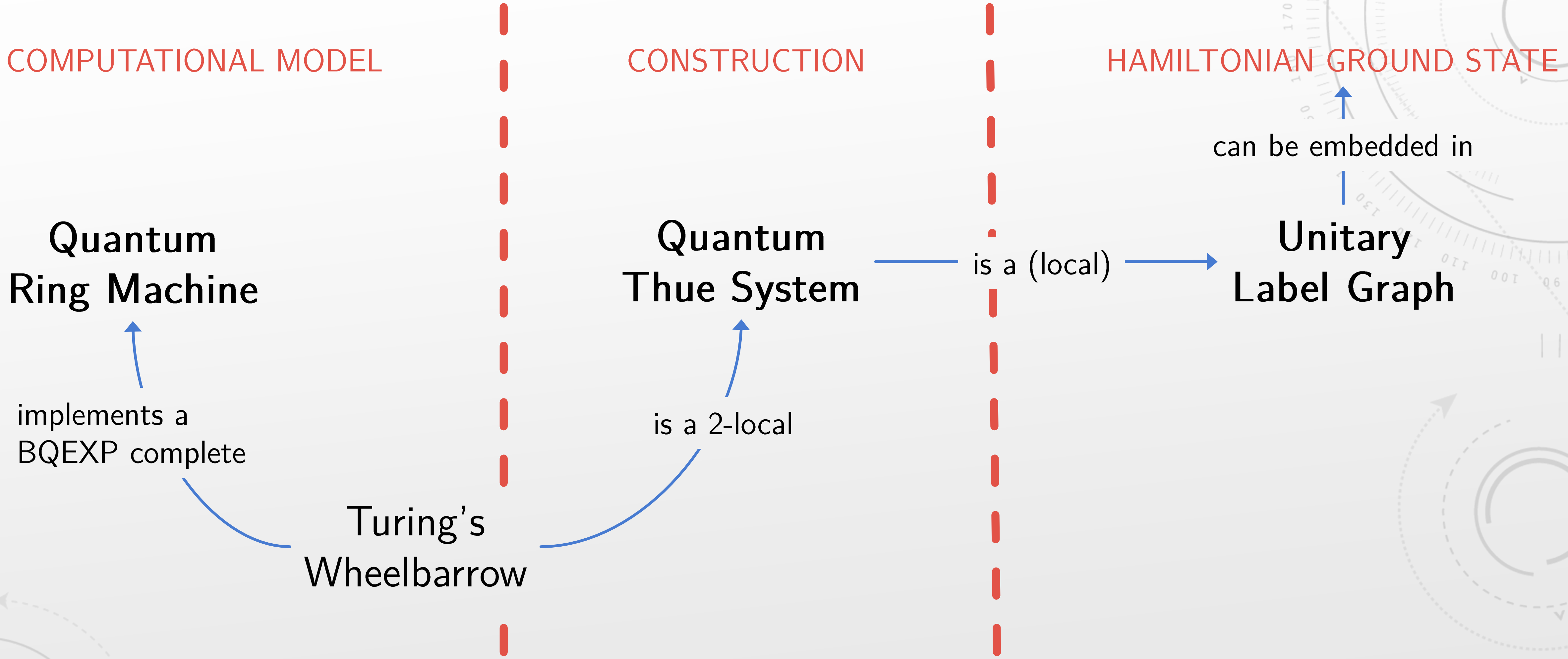


Definition. A ULG is called *semi-classical* if the product of unitaries around any loop is $\mathbb{1}$.

Theorem. Let $\mathbf{H}$ be the Hamiltonian of a semi-classical ULG.
Then $\mathbf{H}$ is *simple*—unitarily equivalent to copies of a graph Laplacian Δ,
i.e. $\exists$ unitary $\mathbf{D}$ such that $\mathbf{H} = \mathbf{D}(\Delta \otimes \mathbb{1})\mathbf{D}^\dagger$.

$\rightarrow$ Spectral Analysis of Hamiltonian

OVERVIEW


COMPUTATIONAL MODEL
CONSTRUCTION
HAMILTONIAN GROUND STATE
can be embedded in
Quantum Ring Machine
Quantum Thue System
is a (local)
Unitary Label Graph
implements a BQEXP complete
is a 2-local
Turing's Wheelbarrow

HAMILTONIAN COMPLEXITY

Theorem.
The local Hamiltonian problem for translationally invariant interactions between neighbouring spins on a chain with local dimension 41 is $\mathsf{QMA}_{\mathsf{EXP}}$-complete.

THANKS!