Strong Converse and Finite Resource Trade-Offs for Quantum Channels

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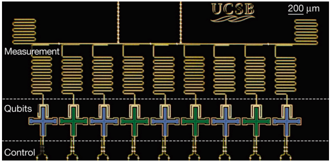
Finite Resource Information Theory

- We are on the verge of engineering small, reliable quantum information processors.

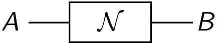


Source: Kelly et al., Nature **519**, 66-69 (2015)

- It is important to understand the fundamental limits for information processing with such small quantum devices.
- We are interested in analytic and easy to evaluate formulas that characterize the trade-off between
  1. the information processing rate (in qubits per use of a resource)
  2. the tolerated error / infidelity
  3. the size of quantum devices / coding block length

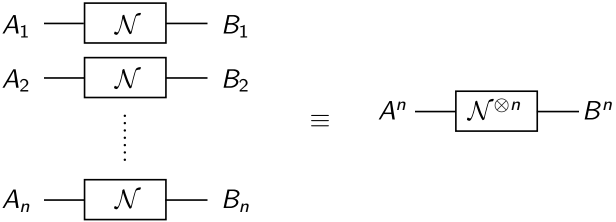
Quantum Coding: Channels

- **Quantum channel**: completely positive trace-preserving linear map $\mathcal{N} \equiv \mathcal{N}_{A \to B}$ from (states on) A to (states on) B.



  Assume A and B are finite-dimensional.

- The channel is memoryless:



Quantum Coding: Encoder and Decoder

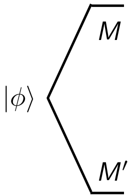
- **Entanglement transmission code** (for $\mathcal{N}^{\otimes n}$):

$$\mathcal{C}_n = \left\{ d_n, \mathcal{E}_n, \mathcal{D}_n \right\} .$$

1. code size d_n:
   - Hilbert spaces M, M', M'' of dimension d_n.
   - maximally entangled state

$$|\phi\rangle_{MM'} = \frac{1}{\sqrt{d_n}} \sum_{i=1}^{d_n} |i\rangle_M \otimes |i\rangle_{M'} .$$

2. encoder $\mathcal{E}_n$: quantum channel from M' to A^n.
3. decoder $\mathcal{D}_n$: quantum channel from B^n to M''.



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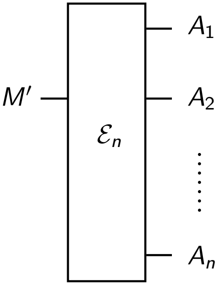

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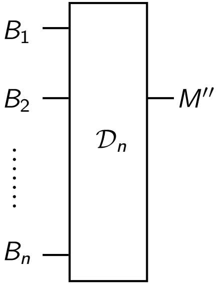
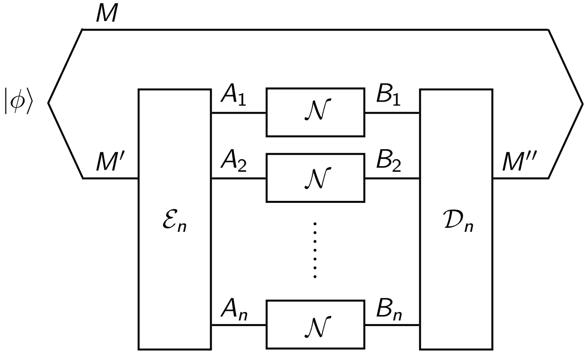


Quantum Coding: Entanglement Fidelity



- **Fidelity** with maximally entangled state:

$$F(\mathcal{C}_n, \mathcal{N}^{\otimes n}) = \operatorname{tr}\Big((\mathcal{D}_n \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}_n)(\phi_{MM'}) \phi_{MM''} \Big) .$$

Achievable Region and Capacity

- A triple (R, n, ε) is achievable on $\mathcal{N}$ if $\exists\ \mathcal{C}_n$ with

$$\frac{1}{n}\log d_n \geq R, \quad \text{and} \quad F(\mathcal{C}_n \mathcal{N}^{\otimes n}) \geq 1-\varepsilon\,.$$

- Boundary of (non-asymptotic) achievable region:

$$\hat{R}(n;\varepsilon,\mathcal{N}) := \max\left\{R : (R,n,\varepsilon) \text{ is achievable on } \mathcal{N}\right\}.$$

- The *quantum capacity*, $Q(\mathcal{N})$, is the rate at which qubits can be transmitted with fidelity approaching one asymptotically.

$$Q_\varepsilon(\mathcal{N}) := \lim_{n\to\infty} \hat{R}(n;\varepsilon,\mathcal{N}), \qquad \varepsilon \in (0,1)$$
$$Q(\mathcal{N}) := \lim_{\varepsilon\to 0} Q_\varepsilon(\mathcal{N})\,.$$

- These are operational quantities: the task of information theory is to relate them to (easy to evaluate) information quantities.

Quantum Capacity Theorem

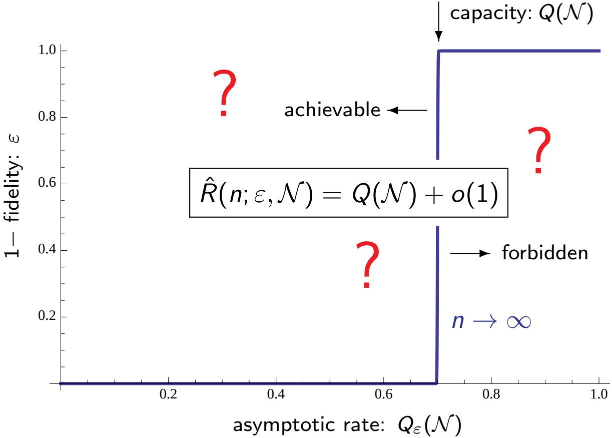
- Barnum, Nielsen and Schumacher (1996-2000) as well as Lloyd, Shor and Devetak (1997-2005) established

$$Q(\mathcal{N}) = \lim_{\ell \to \infty} \frac{1}{\ell} I_c(\mathcal{N}^{\otimes \ell}), \qquad \text{where}$$
$$I_c(\mathcal{N}) := \max_{\rho_A} \{-H(A|B)_\omega\},$$

  and $\omega_{AB} = \mathcal{N}_{A' \to B}(\psi^{\rho}_{A'A})$ for the purification $\psi^{\rho}_{A'A}$ of ρ_A.
- This result is unsatisfactory for several reasons:
  - It is not a single-letter formula, i.e. not easier to compute than the original optimization problem.
  - We need to consider arbitrarily large ℓ in general (Cubitt+'14).
- The formula simplifies for channels which satisfy $I_c(\mathcal{N}^{\otimes \ell}) = \ell I_c(\mathcal{N})$, e.g. for degradable channels like *dephasing channels*.
- But even so, this does not tell us about $\varepsilon > 0$ and finite n.

Capacity and Strong Converse

- Before we consider finite resource trade-offs, we need to fully understand the asymptotic limit $n \to \infty$.
- The first thing we would like to know:



$$\hat{R}(n; \varepsilon, \mathcal{N}) = Q(\mathcal{N}) + o(1)$$

State of the Art

- **Prior to this work, the strong converse property could only be established for some channels with trivial capacity.**
- Morgan and Winter showed that *degradable quantum channels* satisfy a "pretty strong" converse:

$$Q_\varepsilon(\mathcal{N}) = Q(\mathcal{N}) \quad \text{for all } \varepsilon \in \left(0, \frac{1}{2}\right)$$

  (Extending their proof to all $\varepsilon \in (0, 1)$ appears difficult.)
- Strong converse rates are known, for example the entanglement-assisted capacity established via channel simulation (Bennett+'02), or the entanglement cost of a channel (Berta+'13).
- However, they are not tight except for trivial channels.

Result 1: Rains Entropy is Strong Converse Rate

- The *Rains relative entropy* of the channel is defined as

$$R(\mathcal{N}) := \max_{\rho_A} \min_{\sigma_{AB} \in \mathrm{Rains}(A:B)} D\big(\mathcal{N}_{A' \to B}(\psi^{\rho}_{A'A}) \,\big\|\, \sigma_{AB}\big) .$$

Theorem

For any channel $\mathcal{N}$, communication at a rate exceeding $R(\mathcal{N})$ implies (exponentially) vanishing fidelity.

- Key Idea: Consider correlations σ_{AB} that are useless for quantum communication. Classically:

$$C(W) = \max_{P_X} \min_{Q_X, Q_Y} D\big(P_X \times W_{Y|X} \big\| Q_X \times Q_Y\big) = \max_{P_X} I(X : Y) .$$

- A state $\sigma_{AB} \in \mathrm{Rains}(A : B)$ satisfies

$$\mathrm{tr}\left(\phi_{AB}\sigma_{AB}\right) \leq \frac{1}{d} \quad \forall \text{ maximally entangled } \phi_{AB}.$$

- Rains used this set in the context of entanglement distillation (Rains'99).

Result 1: Covariant Channels

- The Rains relative entropy of symmetric channels simplifies.
- Covariance group of the channel $\mathcal{N}$: Group G with unitary representations U_A and V_B such that

$$\mathcal{N}_{A\to B}(U_A(g)(\cdot)U_A^\dagger(g)) = V_B(g)\mathcal{N}_{A\to B}(\cdot)V_B^\dagger(g) \quad \forall g \in G$$

Lemma (Channel Covariance)

Let G be a covariance group of $\mathcal{N}$. Then,

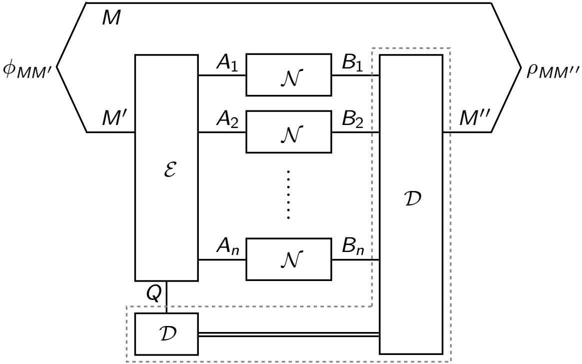
$$R(\mathcal{N}) = \max_{\bar{\rho}_A} \min_{\sigma_{AB}} D\big(\mathcal{N}_{A'\to B}(\psi^{\bar{\rho}}_{AA'}) \,\big\|\, \sigma_{AB}\big)$$

where $\bar{\rho}_A = U_A(g)\bar{\rho}_A U_A^\dagger(g)$, i.e. $\bar{\rho}_A$ is invariant under G.

- Covariance group of $\mathcal{N}^{\otimes n}$ always contains permutations S_n. Thus, we can restrict to permutation invariant states $\bar{\rho}_{A^n}$.
- If the channel is *covariant* with regards to a one-design on A, the optimal state is the maximally entangled state.

Result 1: Assisted Codes

- Remains valid for codes with classical post-processing assistance.



- Includes forward classical communication assistance (all channels).
- Includes two-way communication assistance (covariant channels).
  - Proof via teleportation (Bennett+'96, see also Pirandola+'15).

Example: Dephasing Channels Satisfy Strong Converse

- For all quantum channels we thus have

$$I_c(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q_\varepsilon(\mathcal{N}) \leq R(\mathcal{N}) .$$

Theorem

For generalized dephasing channels $\mathcal{Z}$, we have $I_c(\mathcal{Z}) = R(\mathcal{Z})$.

- The inequalities collapse and $Q_\varepsilon(\mathcal{Z}) = Q(\mathcal{Z})$.
- Includes qubit dephasing channel:

$$\mathcal{Z}_\lambda : \rho \mapsto (1-\lambda)\rho + \lambda Z \rho Z ,$$

with $Q_\varepsilon(\mathcal{Z}_\lambda) = 1 - h(\lambda)$ for all $\varepsilon \in (0,1)$.

- One- or two-way classical assistance does not help.

Result 2: Outer Bounds on Achievable Region

Theorem

If the covariance group of $\mathcal{N}$ is a one-design on A, then

$$\hat{R}(n;\varepsilon,\mathcal{N}) \leq R(\mathcal{N}) + \sqrt{\frac{V_R(\mathcal{N})}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right)$$

- $V_R(\mathcal{N})$ is (Rains) quantum channel dispersion.

$$R(\mathcal{N}) = \min_{\sigma_{AB}\in \mathrm{Rains}(A:B)} D\big(\mathcal{N}_{A'\to B}(\phi_{A'A}) \,\big\|\, \sigma_{AB}\big),$$
$$V_R(\mathcal{N}) := V\big(\mathcal{N}_{A'\to B}(\phi_{A'A}) \,\big\|\, \sigma^*_{AB}\big),$$

- σ^*_{AB} is the minimizer of the channel Rains information,
- $V(\rho\|\sigma) := \mathrm{tr}\left(\rho(\log\rho - \log\sigma)^2\right) - D(\rho\|\sigma)^2$,
- $\Phi^{-1}(\cdot)$ is inverse of cumulative normal distribution function.

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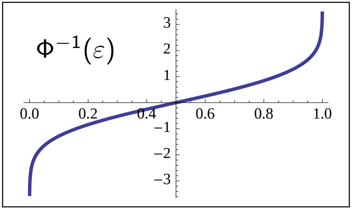


Result 2: Inner Bound on Achievable Region

Theorem

For any quantum channel $\mathcal{N}$, we have

$$\hat{R}(n;\varepsilon,\mathcal{N}) \geq I_c(\mathcal{N}) + \sqrt{\frac{V_c(\mathcal{N})}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right)$$

- $V_c(\mathcal{N})$ is (Hashing) quantum channel dispersion.

$$I_{\mathrm{c}}(\mathcal{N}) = \max_{\rho_A}\left\{D\left(\mathcal{N}_{A'\to B}(\psi^{\rho}_{AA'})\middle\|1_A\otimes\mathcal{N}_{A\to B}(\rho_A)\right)\right\},$$

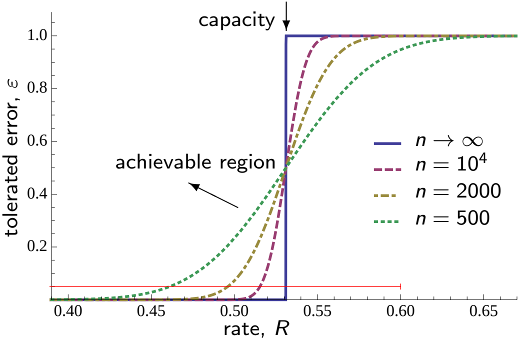
$$V_c(\mathcal{N}) := V\left(\mathcal{N}_{A'\to B}(\psi^{\rho^*}_{A'A})\,\middle\|\,1_A\otimes\mathcal{N}_{A\to B}(\rho^*_A)\right),$$

- ρ^*_A is optimal input state for coherent information.
- This inner bound was independently established by Beigi+'15.
- Sometimes the upper and lower bounds agree...

Example: Qubit Dephasing Channel

- Bounds agree, classical assistance does not help:

$$\hat{R}(n;\varepsilon,\mathcal{Z}_\gamma) = 1 - h(\gamma) + \sqrt{\frac{v(\gamma)}{n}}\,\Phi^{-1}(\varepsilon) + \frac{\log n}{2n} + O\Big(\frac{1}{n}\Big).$$



- Dephasing channel: $\gamma = 0.1$ and fixed fidelity $1 - \varepsilon = 95\%$.
- Corresponds to binary symmetric channel (e.g. Polyanskiy+'10).

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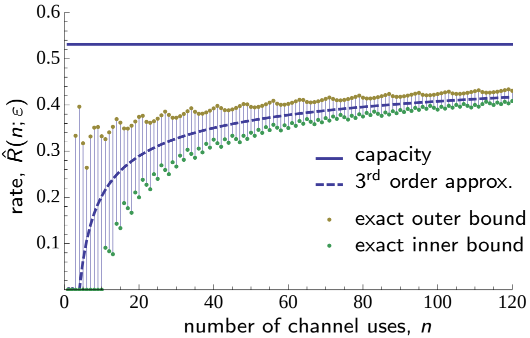

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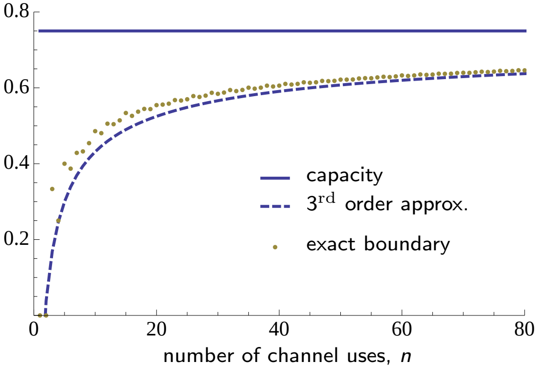
Example: Qubit Erasure Channel

- Erasure channel $\mathcal{E}_\beta : \rho \mapsto (1-\beta)\rho + \beta|k\rangle\langle k|$.
- Bounds agree if we allow two-way classical assistance:

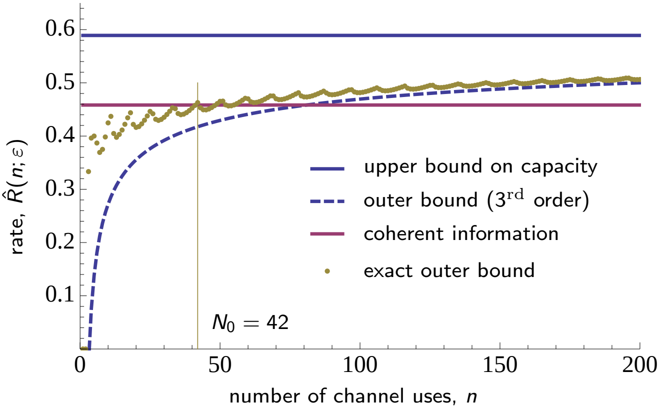
$$\hat{R}(n;\varepsilon,\mathcal{E}_\beta) = 1-\beta+\sqrt{\frac{\beta(1-\beta)}{n}}\Phi^{-1}(\varepsilon)+O\Big(\frac{1}{n}\Big).$$



- Erasure channel: $\beta = 0.25$ and $1-\varepsilon = 99\%$.

Example: Qubit Depolarizing Channel

- Depolarizing channel: $\rho \mapsto (1-\alpha)\rho + \frac{\alpha}{3}\big(X\rho X + Y\rho Y + Z\rho Z\big)$.



- Exact outer bound for $\alpha = 0.0825$ and $\varepsilon = 5.5\%$.
- Inner bounds: unassisted, outer bounds: two-way assisted

Step 1: Arimoto-Type (One-Shot) Converse Bounds

- Consider $\mathcal{C} = \{d, \mathcal{E}, \mathcal{D}\}$ for $\mathcal{N}$ with $F(\mathcal{C}, \mathcal{N}) \geq 1 - \varepsilon$.
- Test if a state is $\phi_{MM''}$, or not:

$$\mathcal{T}(\cdot) = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|, \quad p = \mathrm{tr}\left(\phi_{MM''} \cdot\right).$$

- Let $\rho_{AM} = \mathcal{E}(\phi_{MM'})$. Due to data-processing, we have

$$\min_{\sigma_{BM}} D(\mathcal{N}(\rho_{AM}) \| \sigma_{BM}) \geq \min_{\sigma_{BM}} D\big(\mathcal{T} \circ \mathcal{D} \circ \mathcal{N}(\rho_{AM}) \| \mathcal{T} \circ \mathcal{D}(\sigma_{BM})\big),$$

  for any divergence satisfying data-processing.
- The latter quantity can be bounded using

$$\langle 0|\mathcal{T} \circ \mathcal{D} \circ \mathcal{N}(\rho_{AM})|0\rangle \geq 1 - \varepsilon,$$
$$\langle 0|\mathcal{T} \circ \mathcal{D}(\sigma_{RB})|0\rangle \leq \frac{1}{d}.$$

- Second order: use divergence related to hypothesis testing, D_H^{ε}, and its asymptotic expansion (T+Hayashi'13,Li'14).
- Strong converse: use sandwiched Rényi divergence, $\widetilde{D}_{\alpha}$.

Step 2: Asymptotics for Strong Converse

Lemma

Optimizing over codes we have the following one-shot converse:

$$\hat{R}(1;\varepsilon,\mathcal{N}) \leq \max_{\rho_A} \min_{\sigma_{AB}} \widetilde{D}_\alpha(\mathcal{N}_{A'\to B}(\psi^\rho_{AA'}) \| \sigma_{AB}) + \frac{\alpha \log \frac{1}{1-\varepsilon}}{\alpha - 1}$$

- This yields an upper bound on the ε-capacity:

$$Q_\varepsilon(\mathcal{N}) \leq \limsup_{n\to\infty} \frac{1}{n} \underbrace{\max_{\rho_{A^n}} \min_{\sigma_{A^n B^n}} \widetilde{D}_\alpha\left(\mathcal{N}^{\otimes n}(\psi^\rho_{A^n A'^n}) \middle\| \sigma_{A^n B^n}\right)}_{\widetilde{R}_\alpha(\mathcal{N}^{\otimes n})}.$$

- We can restrict optimization to permutation invariant ρ_{A^n}.
- It remains to show that $\widetilde{R}_\alpha(\mathcal{N})$ satisfies an asymptotic sub-additivity property, i.e. $\widetilde{R}_\alpha(\mathcal{N}^{\otimes n}) \leq n\widetilde{R}_\alpha(\mathcal{N}) + o(n)$.

Step 3: Asymptotic Sub-Additivity

- Employ the fact that $\psi^{\bar{\rho}}_{A^n A'^n}$ is in the symmetric subspace:

$$\psi^{\bar{\rho}}_{AA'} \leq P^{\mathrm{symm}}_{A^n R^n} \leq n^{|A|^2} \int \mathrm{d}\mu(\theta)\, \theta^{\otimes n}_{AR}\,.$$

- The quantum way to restrict to product states in the converse.
- This allows us to show (skipping a few technical steps) that

$$\widetilde{R}_\alpha(\mathcal{N}^{\otimes n}) \leq n\widetilde{R}_\alpha(\mathcal{N}) + O(\log(n)).$$

- Hence, $Q_\varepsilon(\mathcal{N}) \leq \widetilde{R}_\alpha(\mathcal{N})$ for all $\alpha > 1$.
- And, thus, by continuity as $\alpha \to 1$, we find $Q_\varepsilon(\mathcal{N}) \leq R(\mathcal{N})$.
- A more detailed analysis reveals that the fidelity converges exponentially fast to 0 for any $d > R(\mathcal{N})$.

Conclusion

- The (asymptotic) capacity is insufficient to characterize information transmission over quantum channels in realistic settings.
- However, using the channel dispersion, we can characterize the achievable region using only two parameters.
- These approximations agree very well with numerical results already for small instances.

Open Questions:

- Strong converse for all degradable channels.
- Find second order outer bound for general (not only covariant) channels.
- Find better inner bounds for two-way assisted achievable region.
- Consider other important qubit channels, e.g., amplitude damping.