Quantum Minimax Theorem (Extended Abstract)

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Quantum statistical inference is the inference on a quantum system from relatively small amount of measurement data. It covers also precise analysis of statistical error [22], exploration of optimal measurements to extract information [18], development of efficient numerical computation [7]. With the rapid development of experimental techniques, there has been much work on quantum statistical inference [21], which is now applied to quantum tomography [1, 6], validation of entanglement [10], and quantum benchmarks [11, 19]. Thus, quantum statistical inference is one of the core of quantum information science. Many fundamental results in statistical decision theory [25] have been extended to the quantum system. Theoretical framework was originally established by Holevo [12, 13, 14]. Quantum Hunt-Stein theorem [14, 20, 5] and quantum locally asymptotic normality [9, 15] are typical successful examples.

In this context, we show quantum minimax theorem, which is an extension of a well-known result, minimax theorem in statistical decision theory, first shown by Wald [25] and generalized by Le Cam [17]. In order to show how our theorem works, let us take a simplified example in quantum state discrimination, which is also regarded as a discretized version of quantum state estimation. We also explain some concepts in statistical decision theory. (For the meaning of each concept in statistical decision theory, see, e.g., Ferguson [8].) Suppose that Alice has three quantum states

$$\rho_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \rho_2 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \rho_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and randomly chooses one state and sends it to Bob. Bob prepares a POVM to determine which the received quantum state really is. The POVM $\mathbf{M}$ is given by M_1, M_2, M_3, where each element is a three-dimensional positive semidefinite matrix and $M_1 + M_2 + M_3 = I$. When Alice sends i-th state, the probability that Bob obtains the outcome j is given by $p_{\mathbf{M}}(j|i) = \mathrm{Tr}\rho_i M_j$. In this setting, we will find a good POVM. In order to discuss in a quantitative way, we set Bob's loss in the following manner: Bob gets zero if his guess is correct and gets one if his guess is wrong. Using Kronecker's delta, the loss is given by a function of pair (i, j), $w(i, j) = 1 - \delta_{ij},\ i, j = 1, 2, 3$.

Then, the expected loss for Bob conditional to Alice's choice is given by

$$R_{\mathbf{M}}(i) := \sum_{j=1}^{3} w(i, j) p_{\mathbf{M}}(j|i),$$

which is called a *risk function*. For each i, smaller risk is better. Since ρ_1 and ρ_2 are nonorthogonal to each other, there is no POVM that achieves the minimum risk (i.e., zero) for every i. In statistical decision theory, we consider two optimality criteria.

Suppose that Bob has some knowledge on Alice's choice and it is written as a probability distribution, $\pi(1)+\pi(2)+\pi(3) = 1$, which is called a *prior distribution* or shortly *prior*. Then he might consider the average risk,

$$r_{\mathbf{M}}(\pi) := \sum_{i=1}^{3} R_{\mathbf{M}}(i)\pi(i),$$

which is a scalar function of Bob's POVM $\mathbf{M}$. In this setting, there exists a minimizer, which is called a *Bayesian POVM* (a *Bayes POVM*) with respect to π. On the other hand, if Bob has no knowledge on Alice's choice, then he may consider the worst case risk

$$r_{\mathbf{M}}^{SUP} := \sup_i R_{\mathbf{M}}(i),$$

which is again a scalar function of $\mathbf{M}$. There exists a minimizer in this case and it is called a *minimax POVM*.

Bayes POVM and minimax POVM are defined separately and derived from independent optimality criterion. Our main result, quantum minimax theorem gives a deep relation between them. In this example, the theorem says

$$\inf_{\mathbf{M}} r_{\mathbf{M}}^{SUP} = \sup_{\pi} \inf_{\mathbf{M}} r_{\mathbf{M}}(\pi). \tag{1}$$

(We show a much more general version, see Tanaka [24].) One immediate consequence of our theorem is that a minimax POVM is given by a Bayes POVM with respect to a certain prior, which is called a *least favorable prior* [23]. The concept of least favorable priors plays a crucial role in classical Bayesian statistics. Bayesian analysis based on reference priors has been widely recognized among statisticians [2, 3, 4, 16] and the reference prior and its variants are formally defined as least favorable priors.

In the above example, a least favorable prior is $\pi_{LF}(1) = \pi_{LF}(2) = 1/2, \pi_{LF}(3) = 0$. (See Tanaka [24] for obtaining the prior.) Bayes POVM with respect to π_{LF} is

$$M_1 = \frac{1}{2}\begin{pmatrix} 1+1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1-1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \frac{1}{2}\begin{pmatrix} 1-1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1+1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is indeed shown to be minimax.

Our result [24] covers not only quantum state discrimination but also quantum state estimation and quantum state tomography. For any lower-semicontinuous loss function (e.g., squared error, trace distance, fidelity, the Hilbert-Schmidt distance, and the relative entropy), and for every parametric model satisfying regularity conditions (e.g., density matrices and submodel of quantum Gaussian states), we show quantum minimax theorem (a general version of Eq. (1)) and its corollary, e.g., any minimax POVM is also Bayes with respect to π_{LF}. We also show the existence of least favorable priors, which seems to be new in classical Bayesian statistics. It is not a straightforward extension of a classical result. We use no asymptotic techniques like $\rho^{\otimes n}$ and $n \to \infty$ (In modern statistics, it is regarded as a convenient *approximation* for mathematical analysis.).

In addition, if we restrict POVMs to a smaller class of POVMs, e.g., PVMs and its randomization, or separable POVMs over a composite system, still our assertion holds. Thus, our result possibly gives a guideline to many experimenters in quantum physics. In the poster presentation, we will expect fruitful discussions with experimenters.

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