True precision limits in quantum metrology

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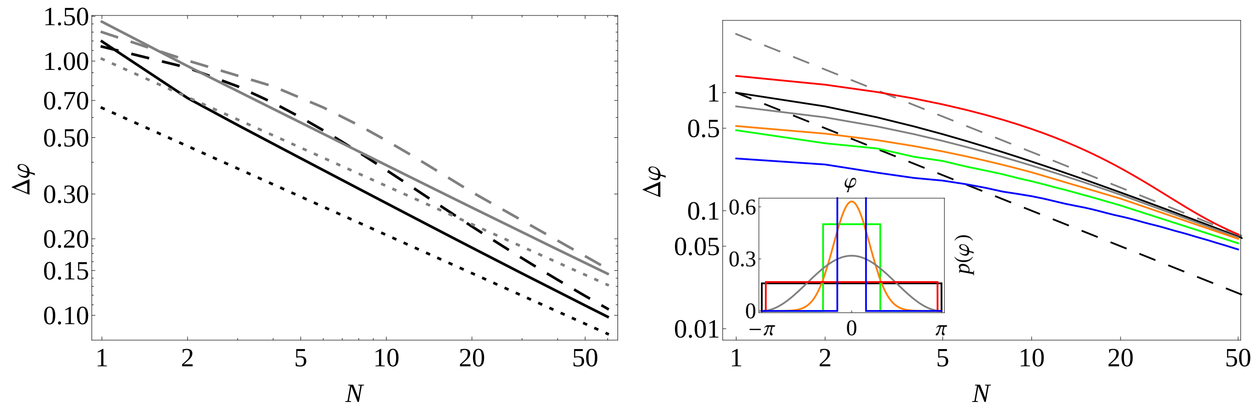
We show that quantification of the performance of quantum-enhanced measurement schemes based on the concept of quantum Fisher information (QFI) yields asymptotically equivalent results as the rigorous Bayesian approach, provided generic uncorrelated noise is present in the setup. At the same time, we show that for the problem of decoherence-free phase estimation this equivalence breaks down and the achievable estimation uncertainty calculated within the Bayesian approach is by a π factor larger than that predicted by the QFI even in the large prior knowledge (small parameter fluctuation) regime, where QFI is conventionally regarded as a reliable figure of merit.

Capability of performing precise measurements is the cornerstone of modern physics. Quantum mechanics provides insight into fundamental limits on the achievable measurement precision that cannot be beaten irrespectively of the extent of any improvements in measurement technology. The best known example is that of the optical phase measurement where difference of phase delays in the arms of interferometer in the absence of decoherence can only be measured up to a precision that scales as $\Delta\varphi \geq 1/N$ where N is the number of photons sent into the setup. This limit is referred as tne Heisenberg limit, as it may be informally viewed as a version of the Heisenberg uncertainty relation adapted to the phase-photon number case. Presence of decoherence, however, which may be due to noise or experimental imperfections, typically prevents quantum-enhanced measurement schemes from reaching the Heisenberg scaling, and it may be demonstrated that for the generic uncorrelated noise processes classically scaling bounds $\Delta\varphi \geq \mathrm{const}/\sqrt{N}$ hold, limiting quantum enhancement to a constant factor precision improvement [1, 2]. Most of the bounds derived in the field of quantum metrology, including the ones mentioned above, are applications of the celebrated Quantum Cramér-Rao (C-R) bound [3] $\Delta\varphi \geq 1/\sqrt{kF}$ which is based on calculation of the Quantum Fisher Information (QFI) $F = \mathrm{tr}\left(\rho_\varphi L_\varphi^2\right)$, where k is the number of independent repetitions of experiment, $\rho_\varphi = \Lambda_\varphi(|\psi_N\rangle\langle\psi_N|)$ is the output state of the channel Λ_φ which imprints value φ of the parameter we want to estimate on the input pure state $|\psi_N\rangle$ of N probes and L_φ is an operator called symmetric logarithmic derivative (SLD) given by equation $\frac{d\rho_\varphi}{d\varphi} = \frac{1}{2}\{\rho_\varphi, L_\varphi\}$. To get the optimal bound one needs now only to optimize QFI over input states. It is known that in principle C-R bound may be saturated by a projective measurement in the eigenbasis of SLD and maximum likelihood estimator in the limit $k \to \infty$.

Practical implications of this last statement are far form obvious, however. The QFI is a point-estimation concept that depends only on the local properties of the state at a given parameter value φ. Saturating the C-R bound may therefore require unrealistically good prior knowledge on the value of the estimated parameter. This is most pronounced by analyzing the behavior of the phase estimation using the N00N states, which are invariant under $2\pi/N$ phase shifts and hence require the prior knowledge of the parameter value to be of the order of $1/N$ as well. Additionally, since L_φ in general depends on φ so can the optimal measurement, and again a significant prior knowledge may be required to perform the optimal measurement. Last but not least, in order to quantify the performance in terms of the *total* resources consumed, i.e. kN, one needs to know the behavior of the required number of repetitions k with the increase of N, which is nontrivial and in general does not lead to analytical formulas.

However, there are also alternative ways of deriving bounds on the performance of quantum-enhanced measurement schemes, that does not suffer from the above mentioned deficiencies, and hence yields the practically achievable precision limits. In particular in the Bayesian approach one explicitly takes into account the prior knowledge about the parameter value, represented by a probability distribution $p(\varphi)$. In this case, we define the average Bayesian error as $\overline{\Delta\varphi} = \sqrt{\int d\varphi \int dx p(\varphi) p_\Pi(x|\varphi)(\varphi - \tilde{\varphi}(x))^2}$ where $p_\Pi(x|\varphi) = \mathrm{tr}\rho_\varphi^N \Pi_x$. Finding the minimal $\overline{\Delta\varphi}$ requires optimization over input state, measurements and estimators which in general is much more demanding than maximization of QFI over input states. Yet, contrary to the QFI case, once the solution is found it yields a the explicit estimation procedure that saturates the minimal average Bayesian error.

In [4] we have shown in which situations Bayesian error is asymptotically equal to the C-R bound. This allowed us to prove that in the case of local uncorrelated decoherence C-R bound is always asymptotically



(a) (b)

FIG. 1: Bayesian error (a) for the flat prior distribution $p(\varphi) = 1/2\pi$ (dashed) vs. bound given by the QFI (solid) as a function of the number of particles for losses (black) and local dephasing (gray) with decoherence parameter $\eta = 0.7$. For comparison ultimate asymptotic QFI based bounds on precision [1, 2] are depicted for losses $\sqrt{\frac{1-\eta}{\eta N}}$ (black, dotted) and dephasing $\sqrt{\frac{1-\eta^2}{\eta^2 N}}$ (gray, dotted). (b) Bayesian error for decoherence-free phase estimation for various prior distribution $p(\varphi)$ all asymptotically converge to π/N formula (gray, dashed). For comparison, $1/N$ C-R bound is given by black dashed line. The shapes of the prior distribution are depicted in the inset.

saturated by Bayesian procedure whereas in the decoherence free case we establish new limits on precision.

In the latter case we have used a general result which states that in the presence of local decoherence and when parameter is encoded unitarily $\Lambda_\varphi = U_\varphi \circ \Lambda$, QFI asymptotically scales linearly with the number of probes in the input state [1, 2]. This allowed us to effecitvely divide entangled input state of N particles into k copies of some other entangled state with smaller number of particles $n = N/k$. By proving that asymptotically $n \overset{N\to\infty}{\to}$ const we obtain that one may effectively think of this case as estimation with some state with fixed number of photons n and number of repetitions k going to infinity, which is the limit in which C-R bound is saturated. Next we have used argument from quantum local asymptotic normality theorem [5] which states that estimation with the state of the form $\rho_\varphi^{\otimes k}$ is asymptotically equivalent to estimation of displacement of some Gaussian state to show that Bayesian error asymptotically is equal to the C-R bound. As an example we plot C-R bound and Bayesian error fig. 1(a) as a function of number of photons N for dephasing and losses and the asymptotic equality between two approaches is clearly seen.

In the case of decoherence-free estimation we have shown that for an arbitrary narrow Gaussian a priori probability distribution Bayesian error scales asymptotically like π/N. Using the fact that this is the same scaling as for flat a priori knowledge [6] we conclude that π/N is the best precision obtained also for all intermediate cases. This however means that irrespectively of a priori knowledge Bayesian precision asymptotically is always given by π/N which is greater than conventional Heisenberg scaling obtained from C-R bound by a factor of π (see fig. 1(b)). Moreover, based on numerical calculations we were able to conjecture that for general unitary evolution $U_\varphi = e^{i\varphi H}$ where H is Hamiltonian with largest and smallest eigenvalues λ_+, λ_- respectively, Bayesian error asymptotically scales as $\pi/N(\lambda_+ - \lambda_-)$, which also differs by a π factor from the C-R bound limit of such case [7].

Additionally we considered also what is the behavior of Bayesian error and C-R bound for collective dephasing which is an example of global decoherence channel - it cannot be decomposed into separate channels acting on each particle only. In such case we obtained that Bayesian precision depends on a priori probability distribution and in general cannot be related to C-R bound. This may be understood intuitively because QFI for such channel is limited from above by some constant, so increasing number of particles at some point does not provide any further advantage. On the other hand if the priori knowledge gives already better precision than C-R bound, it should be no surprise that Bayesian error also would be better.

Eventually we considered also proposed scenarios of beating Heisenberg limit in the decoherence free case, which rely on calculating C-R bound with states with indefinite photon number, constructed as a superposition between low photon number state (for example vacuum) and some high photon number state. We showed that such strategies are ineffective in the Bayesian approach as they cannot give better precision than $\pi/\bar{N}$.

In summary we have proved that in presence of uncorrelated decoherence the asymptotic limits on precision of quantum metrological schemes may be credibly calculated using the QFI approach whereas in the deocherence-free unitary parameter estimation a π factor correction needs to be included irrespectively of the extent of prior knowledge. These observations provide a firm ground for the use of QFI as a sensible figure of merit in analyzing the performance of quantum enhanced metrological protocols based on definite particle number states. In case of strategies employing states with indefinite number of particles the claims remain unchanged in presence of uncorrelated noise, but for the decoherence-free case the Bayesian analysis shows that proposals based on the analysis of the C-R bound which prompted the claims on possibility of sub-Heisenberg estimation strategies are not of much practical use, and the actual Bayesian cost cannot scale better than $\pi/\bar{N}$ where $\bar{N}$ is the average number of particles.

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