Superactivation of quantum nonlocality[*]

C. Palazuelos

January 31, 2013

1 Introduction

The study of quantum nonlocality dates back to the seminal work by Bell ([4]). In this work the author took the apparently metaphysical dispute arising from the previous intuition of Einstein, Podolski and Rosen ([10]) and formulated it in terms of assumptions which naturally lead to a refutable prediction. Given two spatially separated quantum systems, controlled by Alice and Bob respectively and specified by a bipartite quantum state ρ, Bell showed that certain probability distributions $p(a,b|x,y)$ obtained from an experiment in which Alice and Bob perform some measurements x and y in their corresponding systems with possible outputs a and b respectively, cannot be explained by a local hidden variable model (LHVM). Specifically, Bell showed that the assumption of a LHVM implies some inequalities on the set of probability distributions $p(a,b|x,y)$, since then called *Bell inequalities*, which are violated by certain quantum probability distributions produced with an *entangled state*. Though initially discovered in the context of *foundations of quantum mechanics*, violations of Bell inequalities, commonly known as *quantum nonlocality*, are nowadays a key point in a wide range of branches of quantum information science. In particular, nonlocal probability distributions provide the quantum advantage in the security of quantum cryptography protocols ([2], [1]), communication complexity protocols (see the recent review [5]) and in the generation of trusted random numbers ([21]).

The aim of this work is to study the nonlocal properties of bipartite quantum states. Quantum nonlocality is, from its very definition, an effect which occurs on probability distributions $p(a,b|x,y)$. In order to pass from the *probability distribution level* to the *quantum state level*, we say that a bipartite quantum state ρ is *nonlocal* if it can lead to certain quantum probability distributions $p(a,b|x,y)$ in an Alice-Bob scenario violating some Bell inequality. In the case where any probability distribution $p(a,b|x,y)$ produced with the state ρ can be explained by a LHVM, we say that ρ is *local*. Moreover, if we denote by LV_ρ the amount of nonlocality of ρ (we will give a formal definition in the next section), we are mainly interested in two questions.

On the one hand, we want to study *the asymptotic behavior of LV_ρ*. Roughly speaking, if violations of Bell inequalities mean that quantum mechanics is more powerful than classical mechanics, the *amount of Bell violation quantifies how much more powerful it is* (see [13], [14], [15], [6]). Regarding the aim of this work, it is then desirable to have good estimates for a measure LV_ρ of how nonlocal a state ρ is. On the other hand, we will study *the multiplicativity properties of LV_ρ*. It has been shown that in some contexts one can combine two quantum objects to obtain something better than the sum of their individual uses. This effect has been studied in quantum channel theory ([23], [11], [9]) and entanglement theory ([12], [22]). In fact, some of these works show a much stronger behavior called *superactivation*. That is, one can get a quantum effect by combining two objects with no quantum effects. A natural question in our context is whether the quotient $\frac{LV_{\rho^{\otimes k}}}{(LV_\rho)^k}$ can be larger than 1 for a certain natural number k and a certain state ρ and, in that case, how large this quotient can be. Of a special interest is the case where we start by considering a local state ρ (so $LV_\rho = 1$), since in this case we will be dealing with the problem of superactivation of quantum nonlocality.

---

[*]This QIP2013 submission is mainly based on the work [18], but some results from [19] are also included.

2 On the amount of nonlocality of a state ρ

A standard scenario to study quantum nonlocality consists of two spatially separated and non communicating parties, usually called Alice and Bob. Each of them can choose among different observables, labelled by $x = 1, \cdots, N$ in the case of Alice and $y = 1, \cdots, N$ in the case of Bob. The possible outcomes of this measurements are labelled by $a = 1, \cdots, K$ in the case of Alice and $b = 1, \cdots, K$ in the case of Bob. For fixed x, y, we will consider the probability distribution $(P(a, b|x, y))_{a,b=1}^{K}$. Actually, the collection $P = (P(a, b|x, y))_{x,y;a,b=1}^{N,K} \in \mathbb{R}^{N^2 K^2}$ will be also called *probability distribution*.

We say that a probability distribution P is *Classical* if $P(a, b|x, y) = \int_{\Omega} P_{\omega}(a|x) Q_{\omega}(b|y) d\mathbb{P}(\omega)$ for every x, y, a, b, where $(\Omega, \Sigma, \mathbb{P})$ is a probability space, $P_{\omega}(a|x) \geq 0$ for all a, x, ω, $\sum_a P_{\omega}(a|x) = 1$ for all x, ω and the analogous conditions for $Q_{\omega}(b|y)$. We denote the set of classical probability distributions by $\mathcal{L}$. We say that P is *Quantum* if there exist two Hilbert spaces H_1, H_2 such that $P(a, b|x, y) = tr(E_x^a \otimes F_y^b \rho)$ for every x, y, a, b, where $\rho \in B(H_1 \otimes H_2)$ is a density operator and $(E_x^a)_{x,a} \subset B(H_1)$, $(F_y^b)_{y,b} \subset B(H_2)$ are two sets of operators representing POVM measurements on Alice and Bob systems. We denote the set of quantum probability distributions by $\mathcal{Q}$. It is not difficult to see that both $\mathcal{L}$ and $\mathcal{Q}$ are convex sets and, furthermore, that $\mathcal{L}$ is a polytope. The inequalities describing the facets of this set are usually called *Bell inequalities*. As we have explained before, the fact that $\mathcal{L} \subsetneq \mathcal{Q}$ or, equivalently, that there exist some elements $Q \in \mathcal{Q}$ which *violate certain Bell inequalities*, is a crucial point in quantum information theory. We say that a bipartite quantum state ρ is *local* if for all families of POVMs $\{E_x^a\}_{x,a}$, $\{F_y^b\}_{y,b}$, the corresponding probability distribution $Q = (tr(E_x^a \otimes F_y^b \rho))_{x,y;a,b}$ belongs to $\mathcal{L}$. Otherwise, we say that ρ is nonlocal.

In order to separate the sets $\mathcal{L}$ and $\mathcal{Q}$, it is very helpful to slightly extend the notion of Bell inequality. For an arbitrary $M \in \mathbb{R}^{N^2 K^2}$, we consider the quotient $LV(M) = \frac{\omega^*(M)}{\omega(M)}$, where we define $\omega^*(M) = \sup\{|\langle M, Q\rangle| : Q \in \mathcal{Q}\}$ and $\omega(M) = \sup\{|\langle M, P\rangle| : P \in \mathcal{L}\}$ and for every probability distribution P we denote $\langle M, P \rangle = \sum_{x,y;a,b=1}^{N,K} M_{x,y}^{a,b} P(a, b|x, y)$. Note that the existence of Bell violations can be stated by: $LV(M) > 1$ for certain M's. Moreover, $LV(M)$ turns out to be a very useful measure regarding the application of quantum nonlocality in different contexts (see [14], [15]). In order to define a measure of quantum nonlocality for a given state ρ, let us denote $\mathcal{Q}_{\rho}$ the set of all quantum probabilities constructed with the state ρ. Then, for a given element $M \in \mathbb{R}^{N^2 K^2}$, we will denote $LV_{\rho}(M) = \frac{\omega_{\rho}^*(M)}{\omega(M)}$, where $\omega_{\rho}^*(M) = \sup\{|\langle M, Q\rangle| : Q \in Q_{\rho}\}$. Finally, our key object of study is

$$LV_{\rho} := \sup_{N,K} \sup_{M \in \mathbb{R}^{N,K}} LV_{\rho}(M).$$

Since nonlocality usually refers to probability distributions, it is natural to quantify the amount of nonlocality of a state ρ by measuring how nonlocal the quantum probability distributions constructed with ρ can be. LV_{ρ} measures exactly this. Discussing LV_{ρ} in full generality seems difficult. However, some good estimates are known in some specific cases. The following result was proven in [6].

Theorem 1 (Buhrman, Regev, Scarpa, de Wolf)**.**

$$LV_{|\psi_n\rangle} \geq C \frac{n}{(\ln n)^2},$$

where C is a universal constant and $|\psi_n\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |ii\rangle$ denotes the maximally entangled state in dimension n.

3 Results

In order to understand the quantity LV_{ρ} we start by providing an upper bound for it ([19, Section 2]).

Theorem 2. *Given an n-dimensional state ρ, we have*

$$LV_{\rho} \leq \|\rho\|_{S_1^n \otimes_{\pi} S_1^n},$$

where S_1^n denotes the space of $n \times n$ complex matrices with the trace norm and π denotes the projective tensor norm. In particular, $LV_{\rho} \leq n$ for every n-dimensional state ρ.

This means that Theorem 1 is almost tight in the dimension of the state. Actually, this point of view (and further calculations in [19, Section 2]) shows the projective tensor norm as a good candidate to measure the largest Bell violation attainable by a quantum state. This reminds us Rudolph's characterization of entangled states: *a given state ρ is entangled if and only if $\|\rho\|_{S_1^n \otimes_\pi S_1^n} > 1$.* In this sense the previous estimates show a link between quantum entanglement and quantum nonlocality, contrary to the spirit of the most recent results on the topic. The results above make us wonder whether the projective tensor norm of a state $\|\rho\|_{S_1^n \otimes_\pi S_1^n}$ could measure its largest Bell violation LV_ρ up to, maybe, a constant factor. The following theorem shows that this is not the case ([19, Theorem 2.5]).

Theorem 3. *Let $|\psi_n\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n |ii\rangle$ be the maximally entangled state in dimension n. Then*

$$LV_{|\psi_n\rangle} \le D \frac{n}{\sqrt{\ln n}}$$

for a certain universal constant D.

Theorem 1 and Theorem 3 clarify the asymptotic behavior of the largest Bell violation attainable by the maximally entangled state up to the order of the logarithmic factor:

$$C \frac{n}{(\ln n)^2} \le LV_{|\psi_n\rangle} \le D \frac{n}{\sqrt{\ln n}}. \tag{1}$$

In particular, Theorem 3 partially answers the open question posed in [6, Section 1.3] about the possibility of removing the logarithmic factor in Theorem 1. One cannot do that if we restrict to the maximally entangled state.

There is a direct connection between the previous results and the study of the multiplicativity properties of LV_ρ. Indeed, a direct application of Equation (1) allows us to conclude that LV_ρ is a highly non-multiplicative measure. In fact, a careful study of the Bell inequality used in [6] allows us to state the following extreme result ([18, Section V]).

Theorem 4 (Unbounded almost superactivation). *For every $\epsilon > 0$ and $\delta > 0$ we have a state ρ (of a sufficiently high dimension n) verifying that*

$$LV_\rho < 1 + \epsilon \text{ and } LV_{\rho^{\otimes 5}} > \delta.$$

3.1 Superactivation of quantum nonlocality

Theorem 4 leads us to wonder whether it is possible to show superactivation of quantum nonlocality: Does there exist a local state ρ such that $\rho^{\otimes k}$ is nonlocal for some natural number k? Some previous results on superactivation have been obtained in different contexts of quantum nonlocality. A remarkable one was given by Peres ([20]), who showed that superactivation can occur when local pre-processing is allowed on several copies of the state of Alice and Bob (see also [16]). On the other hand, superactivation has been also studied in the context of tensor networks ([7], [8]). More recently, a strong superactivation result was proven in [17] when one is restricted to the particular measurement scenario of two inputs and two outputs per party. However, despite this considerable effort the problem of superactivation of quantum nonlocality has remained open until now.

A careful study of the proof of Theorem 4 shows that one can replace the use of the upper bound in Equation (1) with the use of a classical result by Barret ([3]) to show ([18, Section III]):

Theorem 5 (Superactivation of quantum nonlocality). *There exists an isotropic state $\delta_p = p|\psi_n\rangle\langle\psi_n| + (1-p)\frac{\mathbb{1}}{n^2}$ which is local and such that $\delta_p^{\otimes k}$ is not local for a certain natural number k[1].*

Theorem 5 provides the first example of superactivation of quantum nonlocality in the most general setting. In particular, it answers the recent enhancement of problem 21 posed by Liang ([24]). We must point out that, though Theorem 5 is sharper than Theorem 4 in the sense that it shows a superactivation result, the first theorem is much stronger regarding the amount of violation. Indeed, in Theorem **??** we provide an example of an almost-local state such that five copies of it give an arbitrarily large amount of violation. This is stronger than Theorem 5, where such an unbounded violation can not be obtained by fixing the number of copies k.

---

[1]See [18, Section III] for the precise statement on the values n and p.

References

[1] A. Acin, N. Brunner, N. Gisin, S. Massar, S. Pironio, V. Scarani, Phys. Rev. Lett. 98, 230501 (2007).

[2] A. Acin, L. Masanes, N. Gisin, Phys. Rev. Lett. 97, 120405 (2006).

[3] J. Barrett, Phys. Rev. A 65, 042302 (2002).

[4] J. S. Bell, Physics, **1**, 195 (1964).

[5] H. Buhrman, R. Cleve, S. Massar, R. de Wolf, Rev. Mod. Phys. 82, 665-698 (2010).

[6] H. Buhrman, O. Regev, G. Scarpa, R. de Wolf, IEEE Conference on Computational Complexity 2011: 157-166.

[7] D. Cavalcanti, M. L. Almeida, V. Scarani, A. Acin, Nature Communications 2, 184 (2011).

[8] D. Cavalcanti, R. Rabelo, V. Scarani, Phys. Rev. Lett. 108, 040402 (2012).

[9] T. S. Cubitt, J. Chen, A. W. Harrow, IEEE Trans. Inf. Th., vol. 57, no. 12, pp. 8114-8126, (2011).

[10] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. 47, 777 (1935).

[11] M. B. Hastings, Nature Physics 5, 255 - 257 (2009).

[12] P. Horodecki, M. Horodecki, R. Horodecki, Phys. Rev. Lett. 82 1056-1059, (1999).

[13] M. Junge, C. Palazuelos, Comm. Math. Phys. 306 (3), 695-746 (2011).

[14] M. Junge, C. Palazuelos, D. Pérez-García, I. Villanueva, M.M. Wolf, Phys. Rev. Lett. 104, 170405 (2010).

[15] M. Junge, C. Palazuelos, D. Pérez-García, I. Villanueva, M.M. Wolf, Comm. Math. Phys. 300 (3), 715-739 (2010).

[16] Ll. Masanes, Y.-C. Liang, A. C. Doherty, Phys. Rev. Lett. 100, 090403 (2008).

[17] M. Navascues, T. Vertesi, Phys. Rev. Lett. 106, 060403 (2011).

[18] C. Palazuelos, Phys. Rev. Lett. 109, 190401 (2012).

[19] C. Palazuelos, http://arxiv.org/abs/1206.3695.

[20] A. Peres, Phys. Rev. A, 54, 2685 (1996).

[21] S. Pironio, A. Acín, S. Massar, et al., Nature 464, 1021 (2010).

[22] P. W. Shor, J. A. Smolin, A. V. Thapliyal, Phys. Rev. Lett. 90, 107901 (2003).

[23] G. Smith, J. Yard, Science 321, 1812 - 1815 (2008).

[24] http://qig.itp.uni-hannover.de/qiproblems/21.