



Entanglement Polytopes

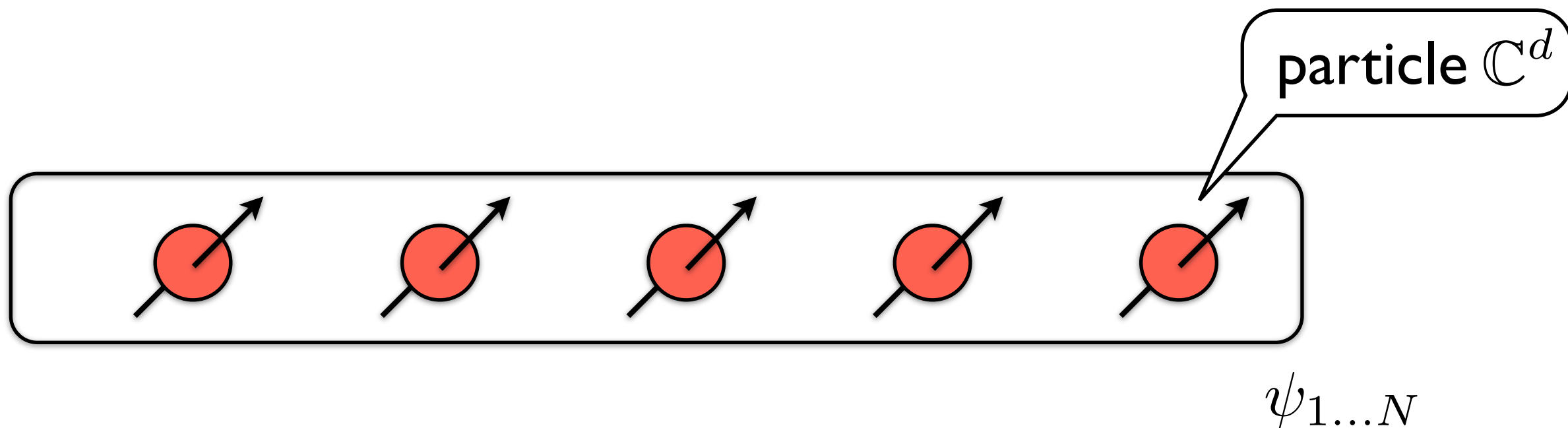
Detecting Genuine Multi-Particle Entanglement by Single-Particle Measurements

Michael Walter

joint work with Matthias Christandl, Brent Doran
(ETH Zürich), and David Gross (Univ. Freiburg)

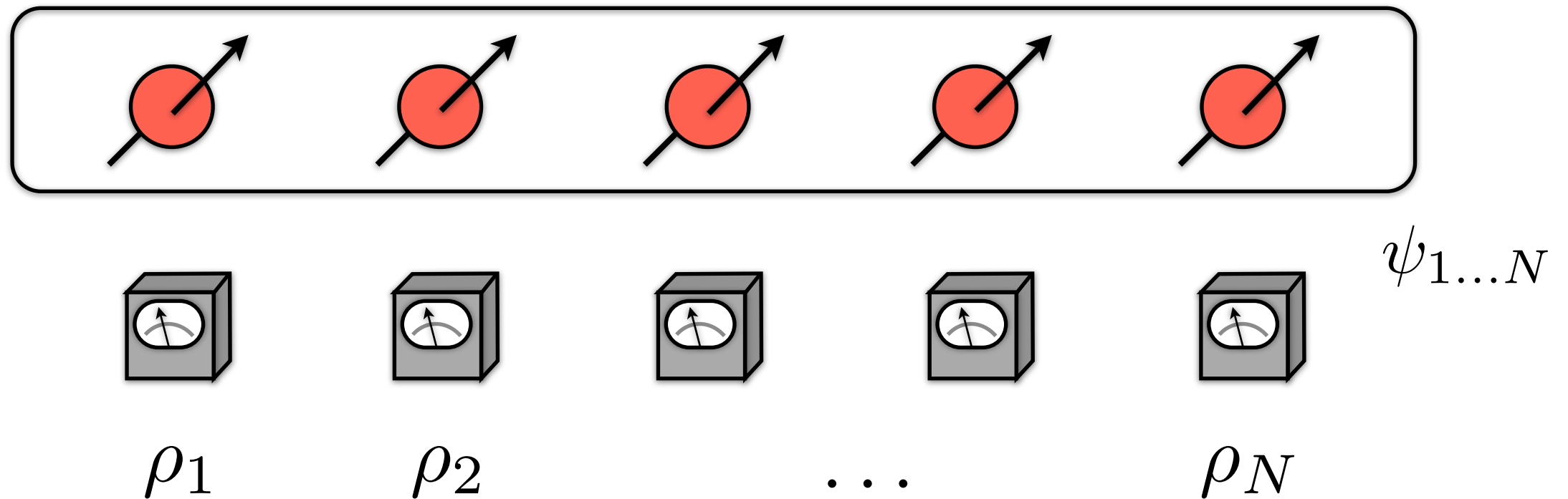


Multi-Particle Entanglement



How entangled is a given multi-particle quantum state $\psi_{1\dots N}$ (prepared in the laboratory)?

Multi-Particle Entanglement via Single-Particle Measurements



What can be said about the multi-particle entanglement of a pure state given its one-body density matrices only?

Multi-Particle Entanglement

Operational approach: equivalent entanglement if related by operations which cannot “create entanglement”

$$\psi_{1\dots N} \longleftrightarrow \psi'_{1\dots N}$$

Multi-Particle Entanglement

Operational approach: equivalent entanglement if related by operations which cannot “create entanglement”



Stochastic Local Operations and Classical Comm.:

- non-zero success probability
- convenient mathematical characterization:

$$\psi'_{1\dots N} = (A_1 \otimes \dots \otimes A_N) \psi_{1\dots N}$$

Dür, Vidal &
Cirac (2000)

invertible local operators

Multi-Particle Entanglement

Entanglement class of a pure state $\psi_{1\dots N}$:

$$\mathcal{C}_\psi = \left\{ \psi' \overset{\text{SLOCC}}{\longleftrightarrow} \psi \right\} = G \cdot |\psi\rangle\langle\psi| \subseteq \mathbb{P}(\mathcal{H})$$

$$G = SL(d) \times \dots \times SL(d)$$

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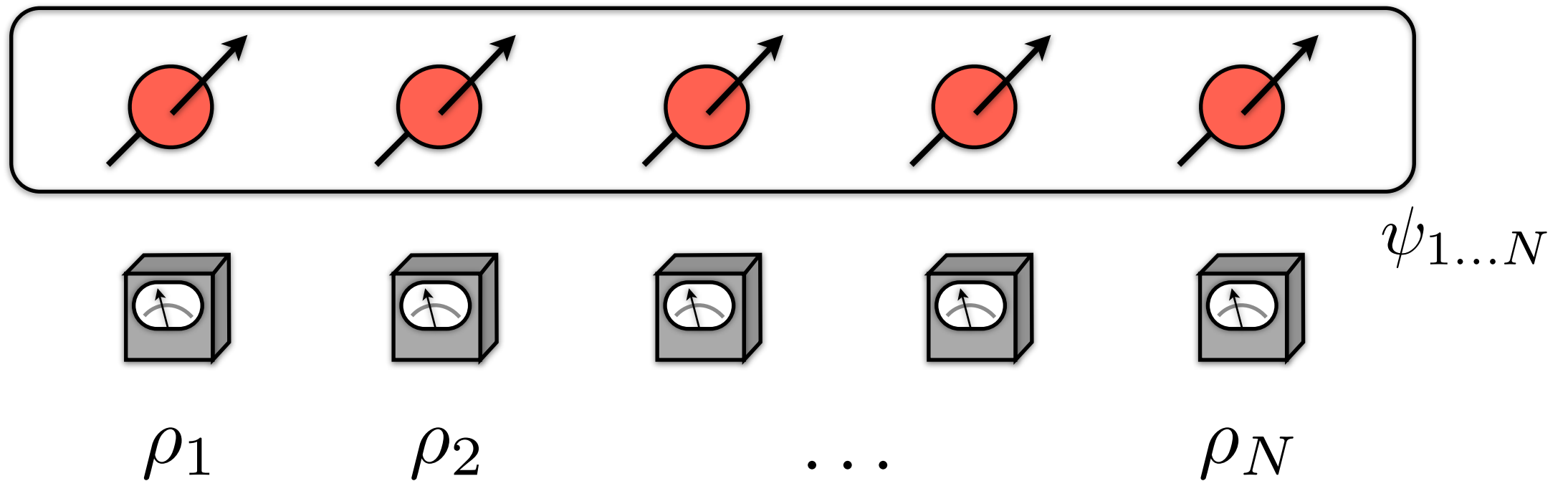
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Three Qubits: six classes (GHZ, W, bi-separable, product)

Generically: infinitely many classes, labeled by $\exp(N)$
many continuous parameters

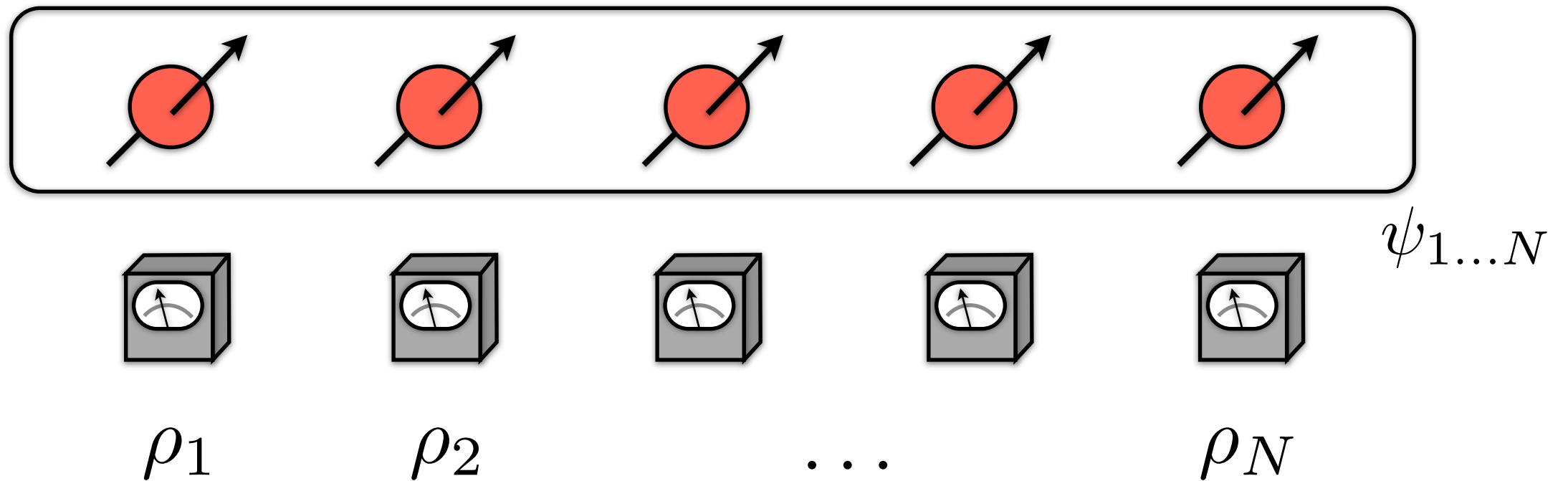
Single-Particle View of Multi-Particle Entanglement



What can be said about the entanglement class of a pure state given its one-body density matrices only?

What are the possible $\rho_1, \dots, \rho_N$ for states in a given entanglement class?

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only depends on local eigenvalues

Entanglement Polytopes



$$\Delta_{\mathcal{C}} = \{ \vec{\lambda} = (\vec{\lambda}_1, \dots, \vec{\lambda}_N) \text{ for } \psi \in \overline{\mathcal{C}} \}$$

eigenvalues of
 $\rho_1, \dots, \rho_N$

Entanglement Polytopes



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entanglement
polytope

Our Main Result:

- convex polytope!
- finite hierarchy
- algorithm to compute using computational invariant theory (difficult)

using results from Brion (1987),
Kempf & Ness (1979)
algebraic geometry / GIT

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cf. Quantum Marginal Problem

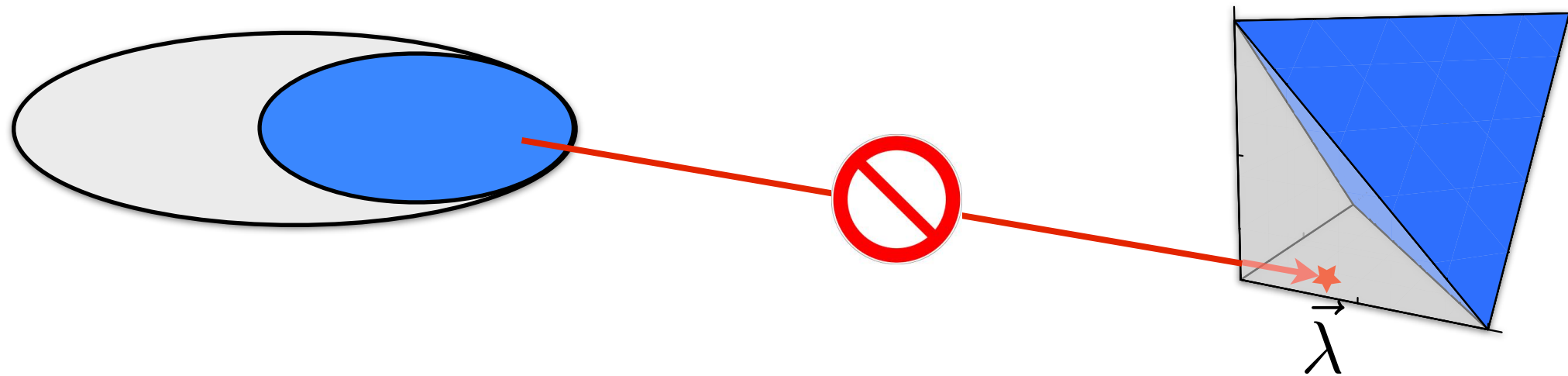
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Christandl-Mitchison (2004)

Klyachko (2004)

Daftuar-Hayden (2004)

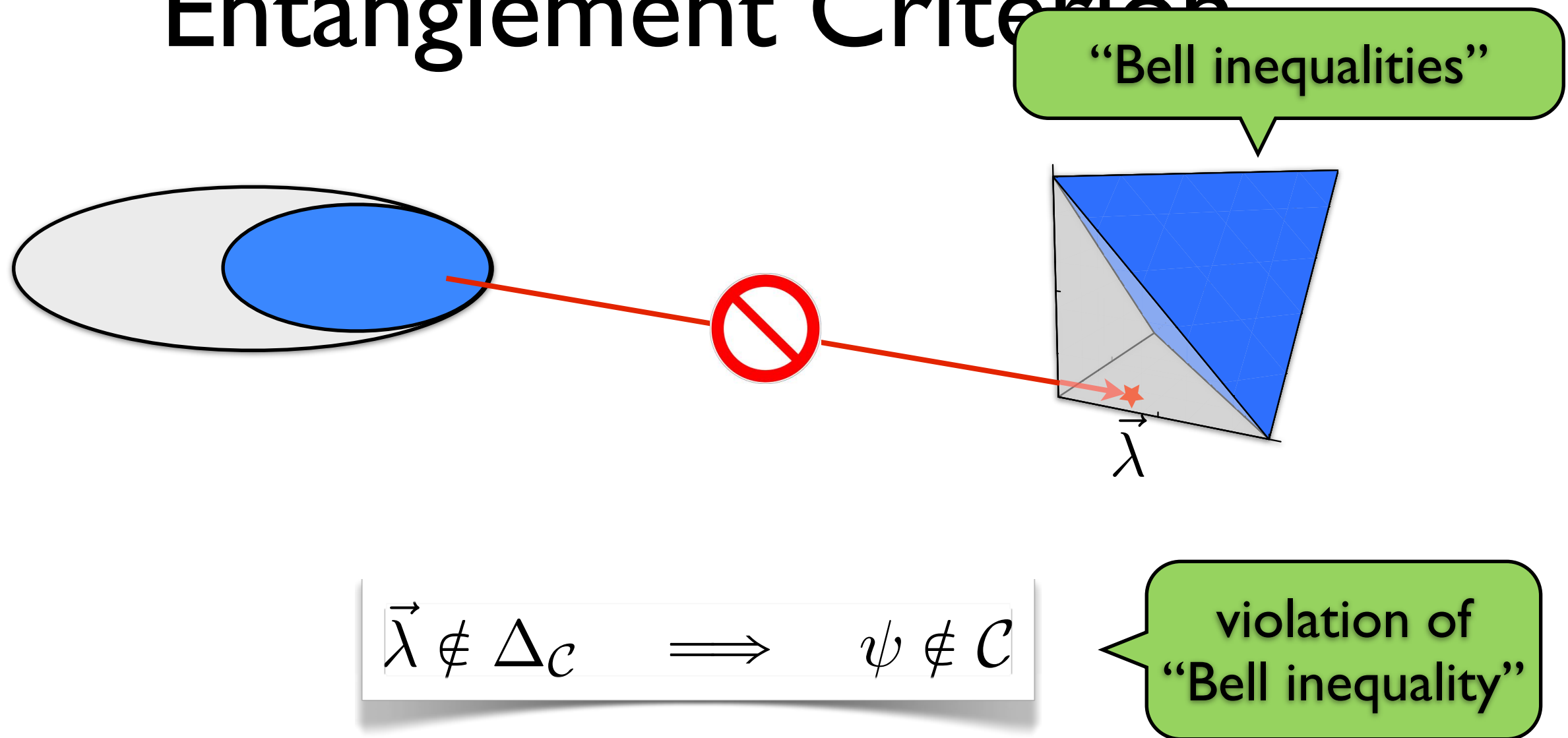
Entanglement Criterion



$$\vec{\lambda} \notin \Delta_{\mathcal{C}} \implies \psi \notin \mathcal{C}$$

- efficient, requires only linearly many measurements
- robust against experimental noise ($\psi \approx \text{pure}$)

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Three Qubits

$$N = 3$$
$$\mathcal{H} = \mathbb{C}^2$$

Entanglement classes:

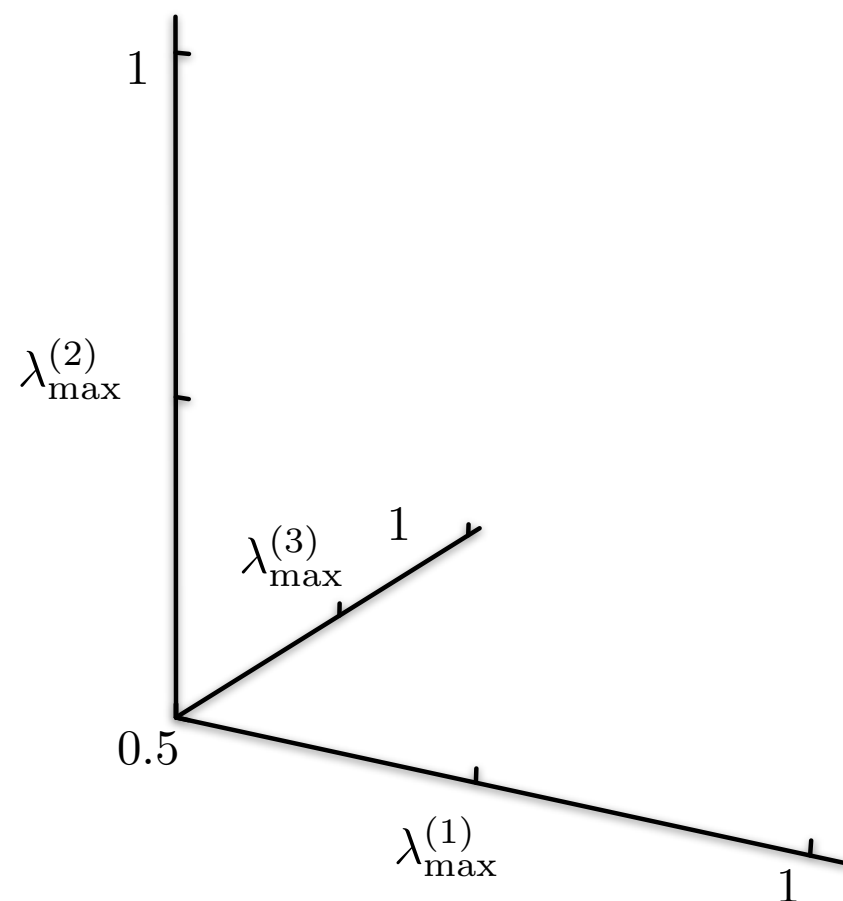
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$$|W\rangle = |100\rangle + |010\rangle + |001\rangle$$

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$$|B2\rangle, |B3\rangle$$

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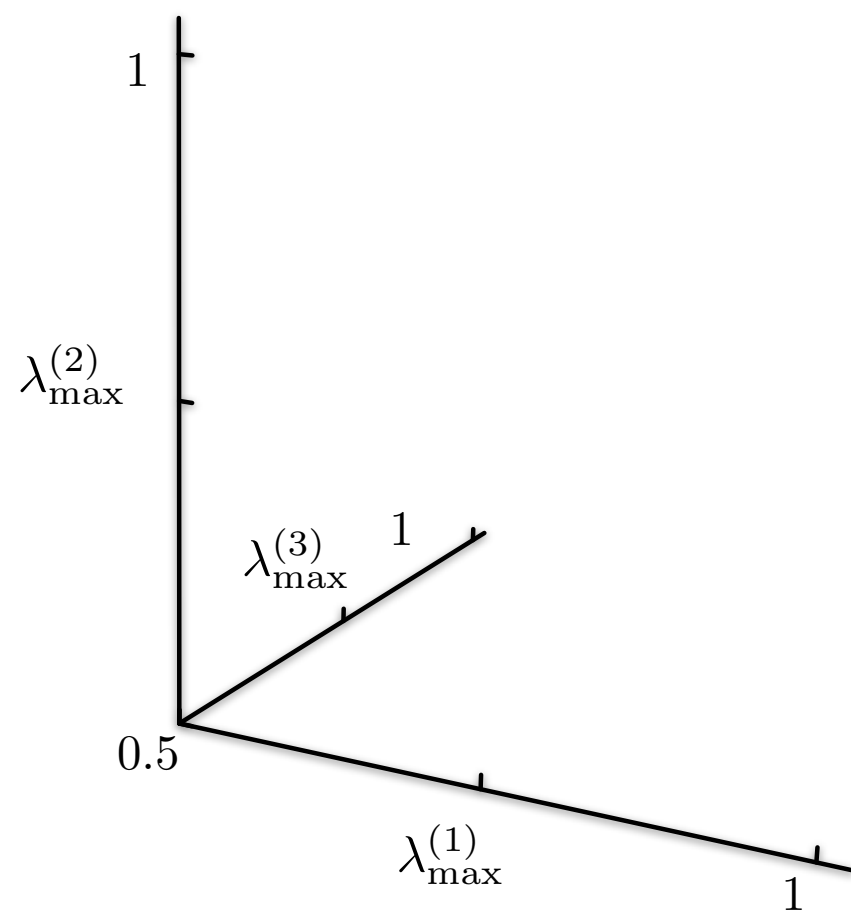


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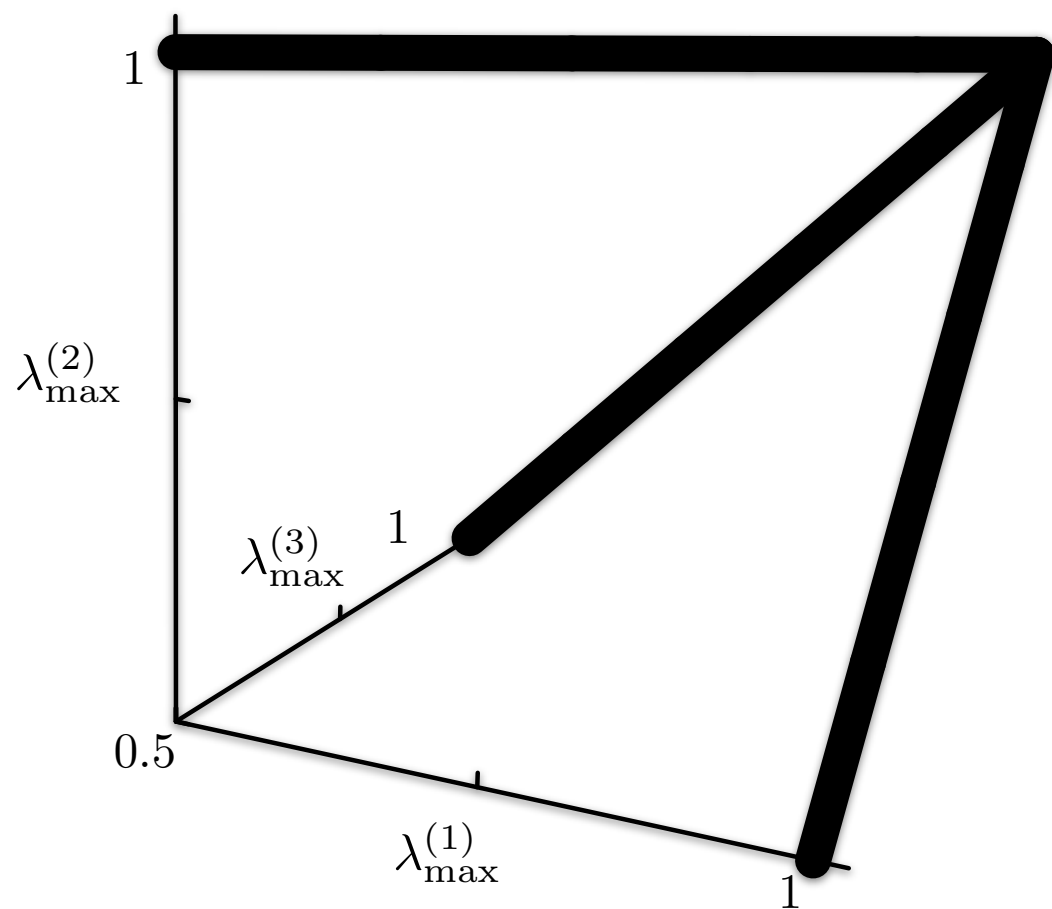
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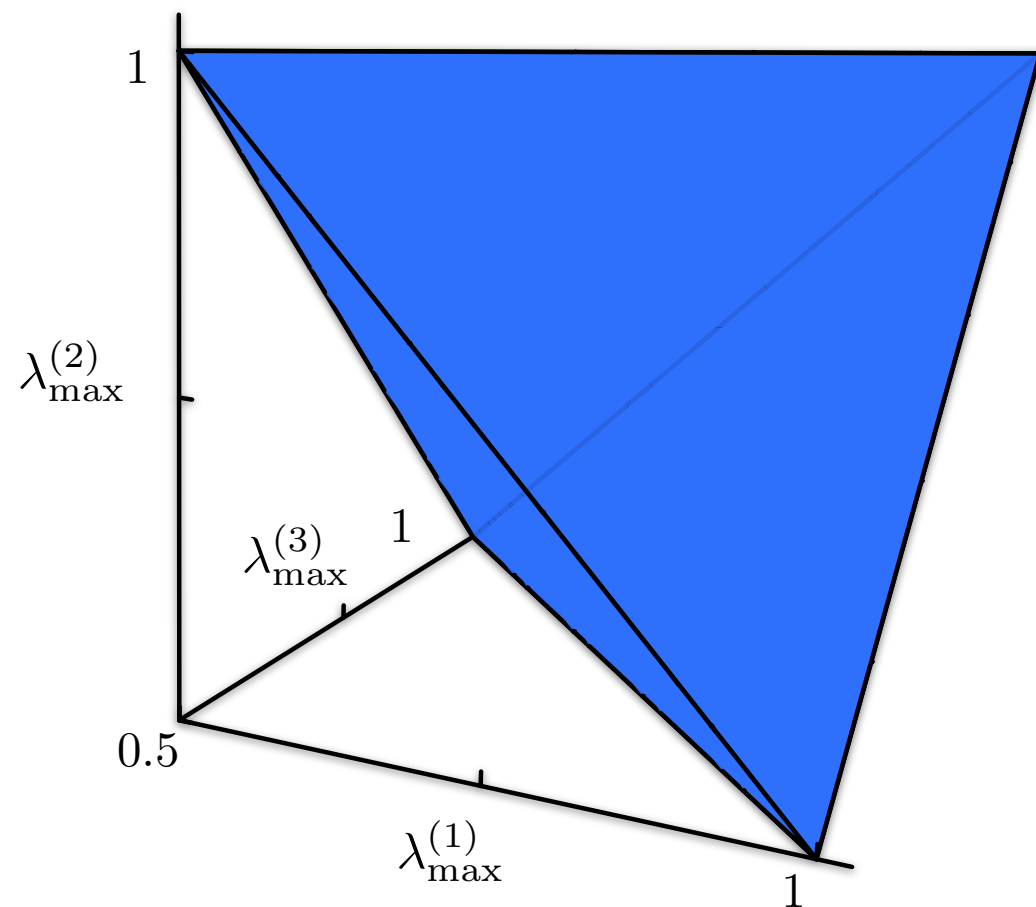
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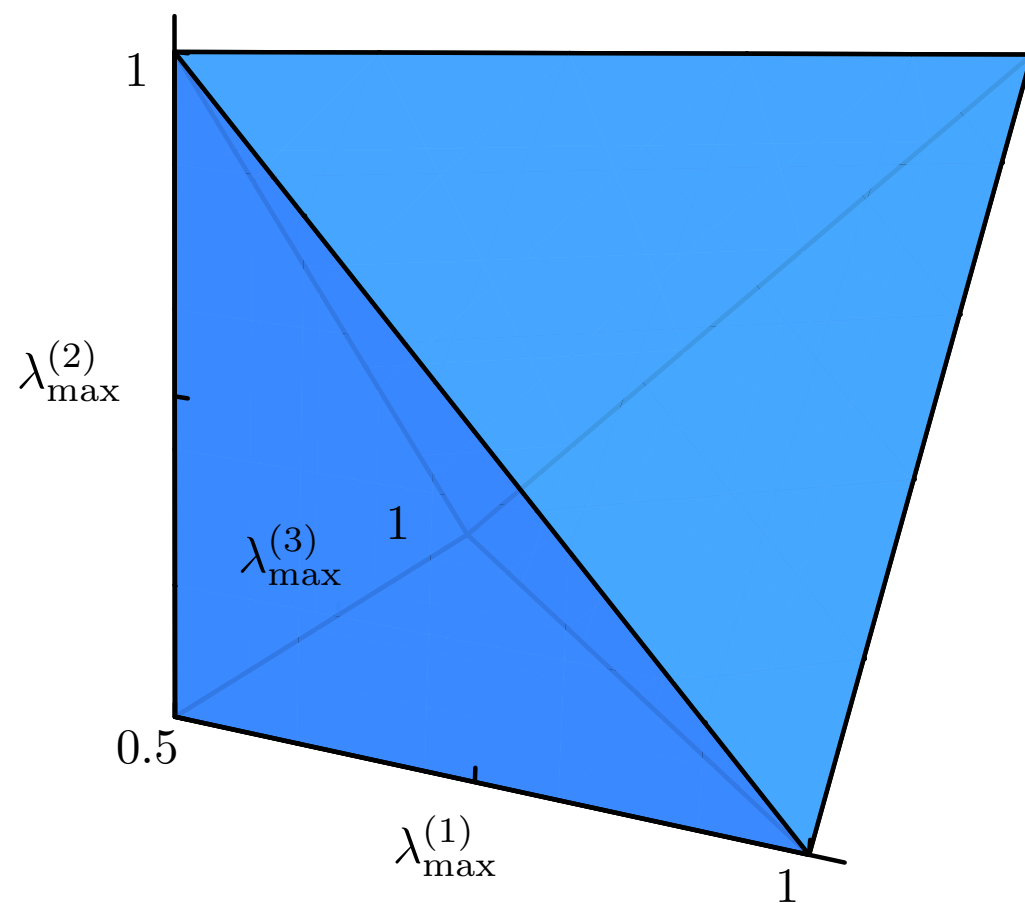
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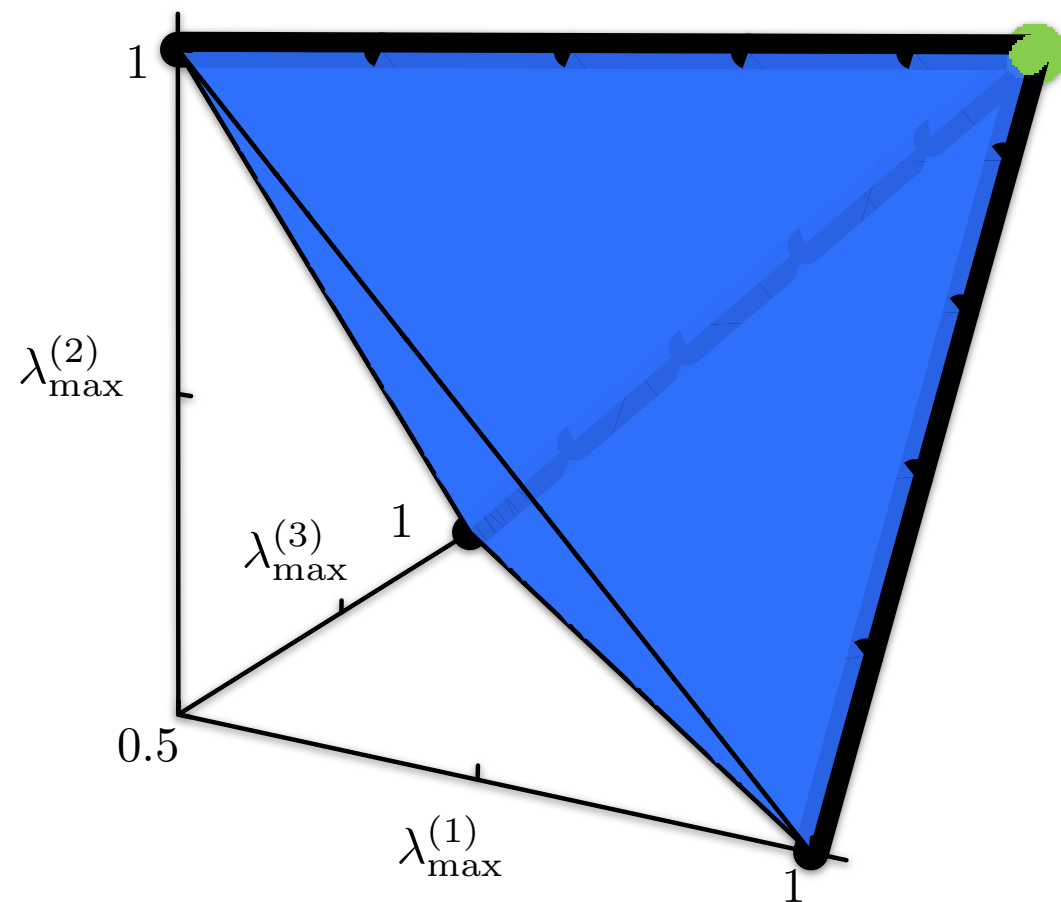
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Lower pyramid is
witness for GHZ class!

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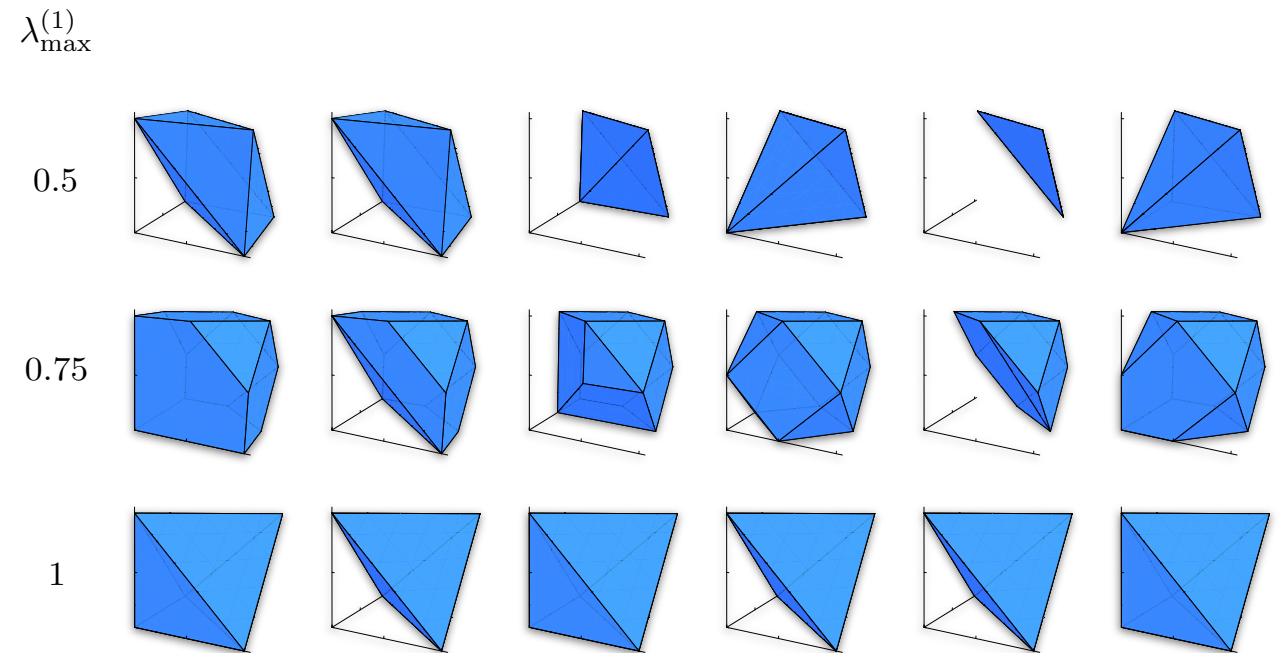
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Further Examples

- Four Qubits: six non-trivial entanglement polytopes (3d sections)



- Bosonic qubit systems, fermionic system, ...

$$\text{Sym}^N \mathbb{C}^2$$

$$\wedge^3 \mathbb{C}^6$$

<http://www.itp.phys.ethz.ch/people/waltemic/polytopes>

- Genuine multi-particle entanglement:

$$\Delta \not\supseteq \bigcup_{\text{bipartition } S:S^c} \Delta_S \times \Delta_{S^c}$$

Invariant Theory

$$\mathcal{H} = \mathbb{C}^d \otimes \dots \otimes \mathbb{C}^d$$
$$G = SL(d) \times \dots \times SL(d)$$

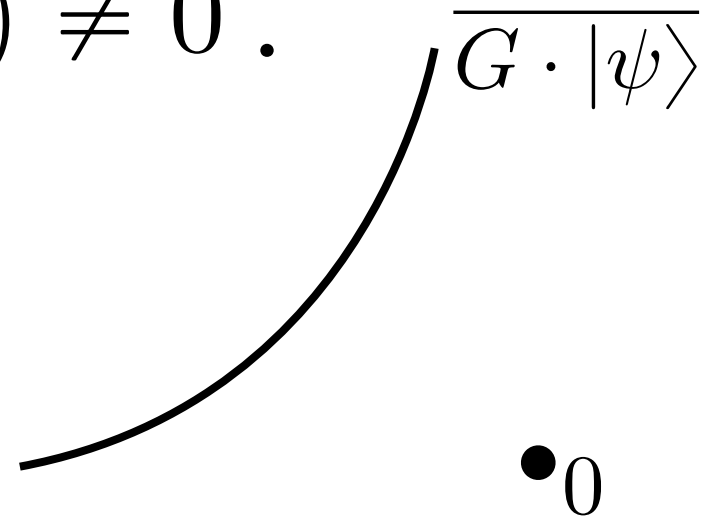
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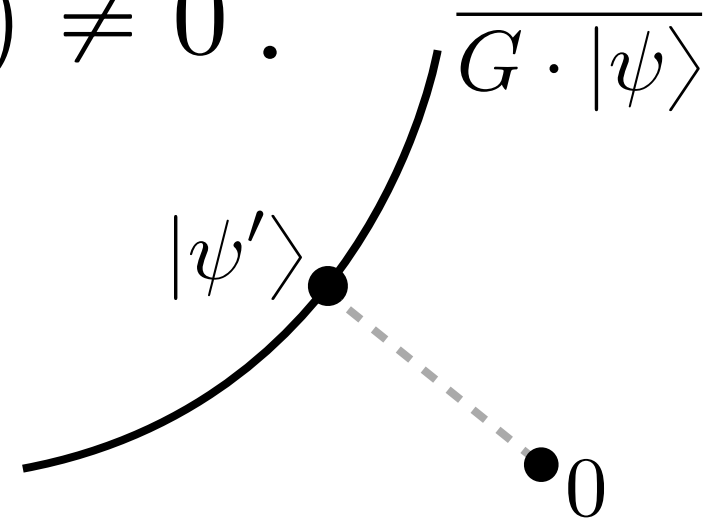
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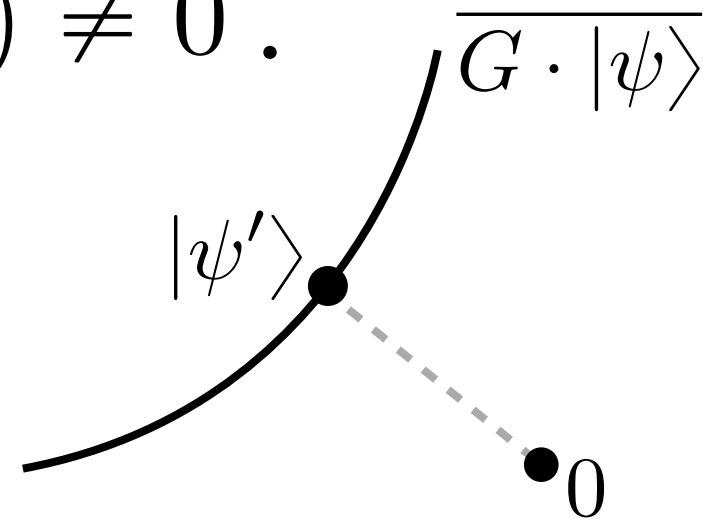
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Let $|\psi'\rangle$ be a vector of minimal length. Then

$$0 = \left. \frac{d}{dt} \right|_{t=0} \|e^{tX} \cdot |\psi'\rangle\|^2 = 2\langle\psi'|X|\psi'\rangle$$

for all traceless local observables X .

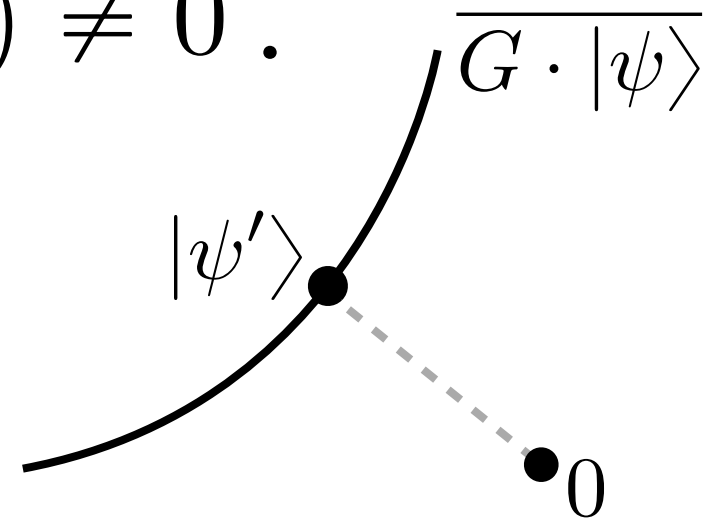
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Thus ψ' is locally maximally mixed!

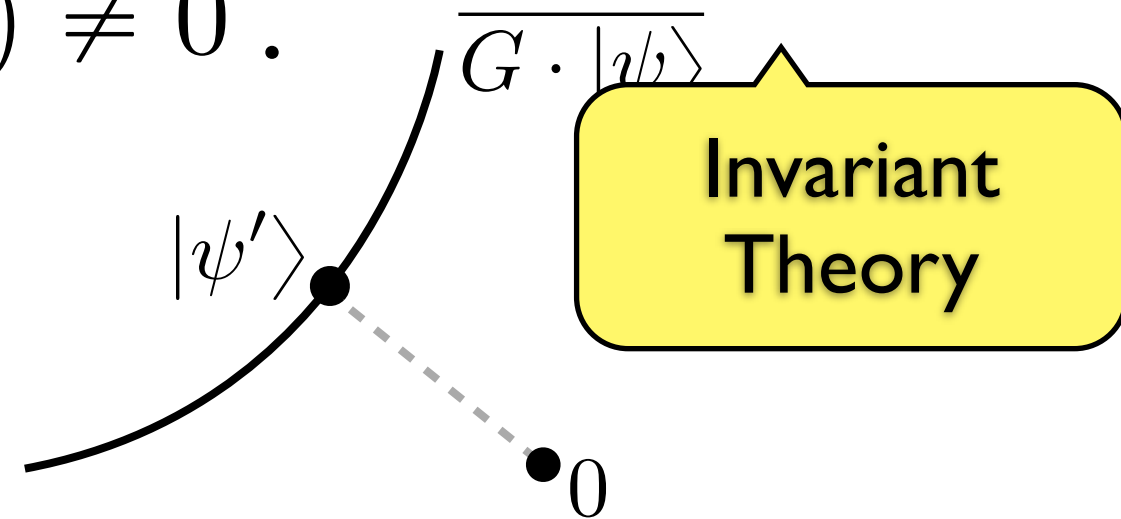
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The converse is also true!

for all traceless local observables X .

Kempf & Ness (1979)
Klyachko (2004)

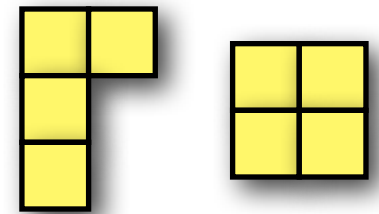
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Local eigenvalues

Entanglement Polytopes and Covariants

Covariant: G -equivariant homogeneous polynomial function

$$\Phi: \mathcal{H} \rightarrow V_\lambda$$

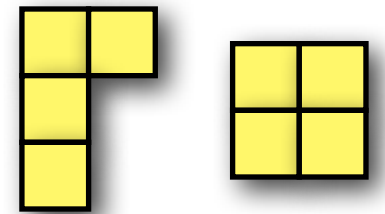


Irreducible G -representation
with highest weight λ

Entanglement Polytopes and Covariants

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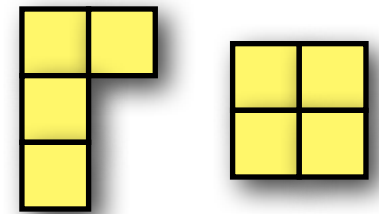


Choose a finite set of generators Φ_k with highest weight λ_k and degree d_k .

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Theorem: The entanglement polytope of an entanglement class $\mathcal{C}_\psi$ is given by

proof via
Brion (1987)

$$\Delta_\psi = \text{conv}\{\lambda_k/d_k : \Phi_k(\psi) \neq 0\}$$

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computational
invariant theory

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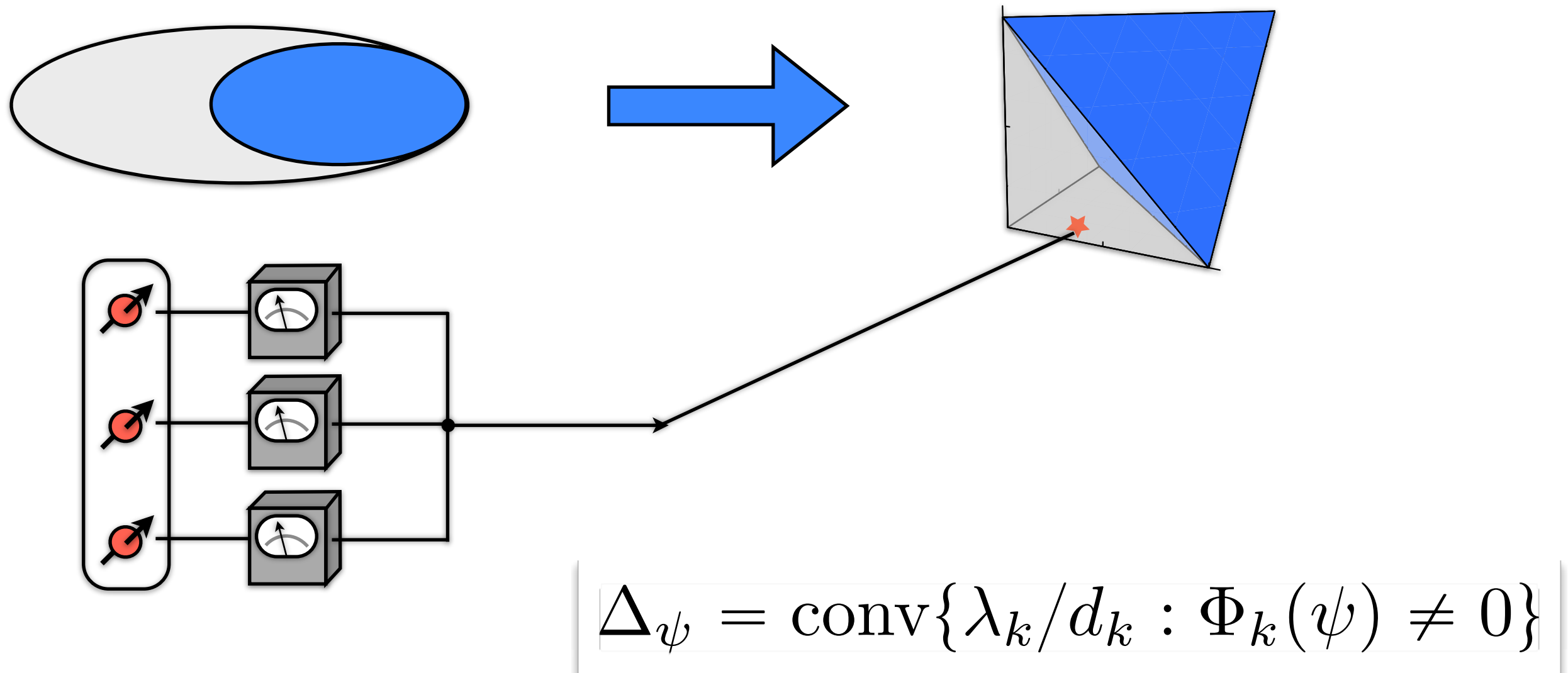
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Summary

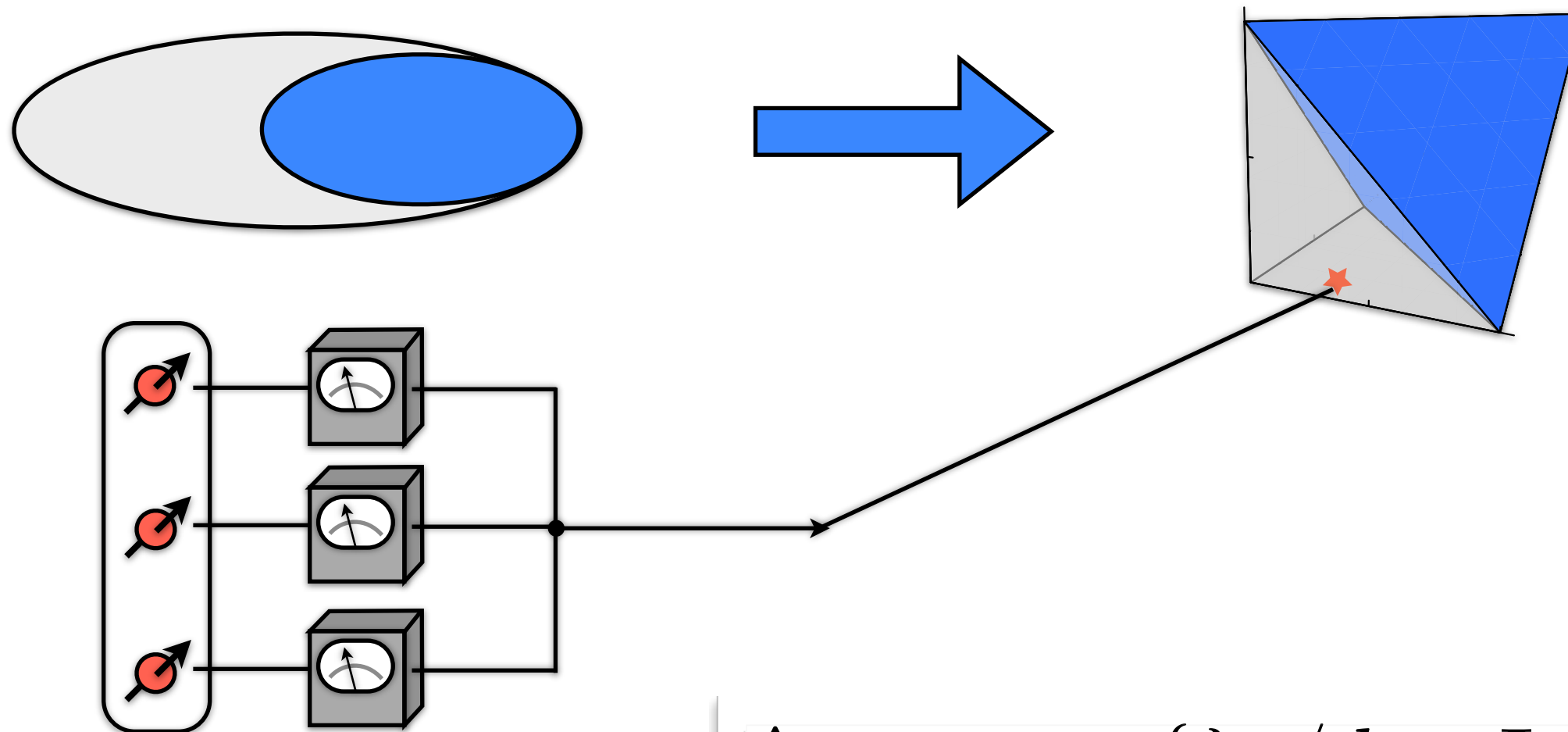
arXiv:1208.0365



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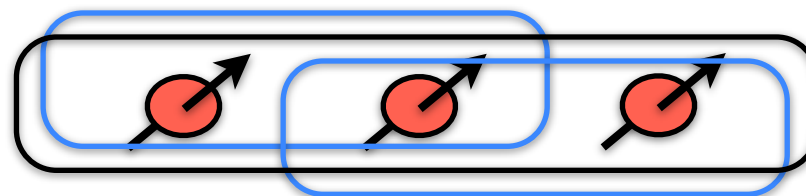
Thank you!

arXiv:1208.0365



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See also Burak's talk on "Recoupling Coefficients and Quantum Entropies"!



Thursday 9:55