Relative entropy and squashed entanglement

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Entanglement and entanglement measures

- Entanglement is an important concept and resource. It is the most outstanding non-classical feature of compound states that can’ t be expressed as mixture of product states.

- Entanglement measures: To understand entanglement, we need entanglement measures with good properties.

  - operational ones: E_c, E_d, K_d
  - : E_f, E_r, E_{sq}, E_n …
  - abstract ones: …

Squashed entanglement and relative entropy of entanglement

- Squashed entanglement (Tucci ' 99, ' 02; Christandl & Winter ' 04)

$$E_{sq}(\rho_{AB}) := \inf\left\{\frac{1}{2}I(A;B|E)_\rho : \rho_{ABE} \text{ is an extension of } \rho_{AB}\right\},$$

where $I(A;B|E)_\rho$ is the quantum conditional mutual information of ρ_{ABE},

$$I(A;B|E)_\rho := S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E)$$

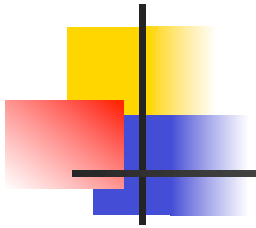
- It has many nice properties (monogamy, additivity, etc) and operational meaning (Koashi & Winter ' 04; Christandl & Winter ' 04 ; Devetak & Yard ' 08)

- Relative entropy of entanglement (Vedral et al ' 97, ' 98)

$$E_r(\rho_{AB}) := \min_{\sigma_{AB}\in \mathrm{SEP}} D(\rho\|\sigma) \text{ with } D(\rho\|\sigma) = \mathrm{Tr}(\rho(\log\rho - \log\sigma))$$

- Regularized version admits operational meaning. (Brandao & Plenio ' 08, ' 10)

$$E_r^\infty(\rho_{AB}) := \lim_{n\to\infty} \frac{1}{n} E_r(\rho_{AB}^{\otimes n})$$

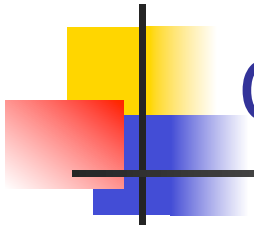


- Post-measurement relative entropy of entanglement, with respect to restricted measurement classes (Piani ' 09)

$$E_{r,\mathrm{M}}(\rho_{AB}) := \inf_{\sigma\in\mathrm{SEP}} \sup_{\mathcal{M}\in\mathrm{M}} D\big(\mathcal{M}(\rho_{AB})\|\mathcal{M}(\sigma_{AB})\big)$$

Here M is a class of measurement, such as

LO, 1-LOCC, LOCC, SEP and PPT.



Outline

- Results:
  1. Monogamy relation for relative entropy of entanglement
  2. Commensurate lower bound for squashed entanglement
  3. Properties of $E_{r,\mathrm{M}}(\rho_{AB})$: asymptotic continuity and evaluation on maximally entangled states and pure states.
  4. Comparisons between entanglement measures.

- Proofs (of 1 and 2)

Monogamy relation for relative entropy of entanglement

For an entanglement measure f, one would expect the monogamy

$$f(\rho_{1:23}) \geq f(\rho_{1:2}) + f(\rho_{1:3})$$

This is true for E_{sq} but fails for E_r (along with most other EMs).

Counterexample: anti-symmetric states! (Christandl, Schuch, Winter '10)

Properly weakened monogamy relation:

Theorem 1 For every tripartite quantum state ρ_{ABE}, we have

$$E_r(\rho_{B:AE}) \geq E_{r,\text{1-LOCC}}(\rho_{AB}) + E_r^{\infty}(\rho_{BE}),$$

$$E_r^{\infty}(\rho_{B:AE}) \geq E_{r,\text{1-LOCC}}^{\infty}(\rho_{AB}) + E_r^{\infty}(\rho_{BE}).$$

Commensurate lower bound for squashed entanglement

- Squashed entanglement is faithful. (Brandão, Christandl, Yard ’ 10)

$$E_{sq}(\rho_{AB}) > 0 \quad \Leftrightarrow \quad \rho_{AB} \text{ entangled}$$

The main result of the proof is the following 1-LOCC trace-norm bo

$$E_{sq}(\rho_{AB}) \geq \frac{1}{16 \ln 2} \min_{\sigma_{AB} \in \text{SEP}} \|\rho_{AB} - \sigma_{AB}\|^2_{\text{1-LOCC}}$$

$$\|\rho_{AB} - \sigma_{AB}\|_{\text{1-LOCC}} := \sup_{\mathcal{M} \in \text{1-LOCC}} \|\mathcal{M}(\rho_{AB}) - \mathcal{M}(\sigma_{AB})\|$$

Commensurate lower bound for squashed entanglement

- We provide a 1-LOCC relative entropy lower bound for E_{sq}

Theorem 2 For every quantum state ρ_{AB}, we have

$$E_{sq}(\rho_{AB}) \geq \frac{1}{2} E^{\infty}_{r,1\text{-LOCC}}(\rho_{AB}) \geq \frac{1}{2} E_{r,1\text{-LOCC}}(\rho_{AB}).$$

- Strong subadditivity: $S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E) \geq 0$ (Lieb, Ruskai '73)

$$S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E) \geq E_{r,1\text{-LOCC}}(\rho_{AB}).$$

$$E_{r,1\text{-LOCC}}(\rho_{AB}) := \inf_{\sigma \in \text{SEP}} \sup_{\mathcal{M} \in 1\text{-LOCC}} D\big(\mathcal{M}(\rho_{AB}) \| \mathcal{M}(\sigma_{AB})\big)$$

(note:)

Commensurate lower bound for squashed entanglement

About the new bound:

- Recovering the 1-LOCC trace-norm bound:
  applying Pinsker's inequality $D(\rho\|\sigma) \geq \frac{1}{2\ln 2}\|\rho-\sigma\|_1^2$,
  we are able to recover the 1-LOCC trace-norm bound (with slightly better constant factor) :

$$E_{sq}(\rho_{AB}) \geq \frac{1}{4\ln 2} \min_{\sigma_{AB}\in \mathrm{SEP}} \|\rho_{AB}-\sigma_{AB}\|_{1\text{-LOCC}}^2$$

- It is asymptotically normalized: for maximally entangled state Φ_d and pure state ψ_{AB} ,

$$E_{r,1\text{-LOCC}}(\Phi_d) = \log(d+1) - 1$$

$$E^{\infty}_{r,1\text{-LOCC}}(\psi_{AB}) = S(\mathrm{Tr}_B\,\psi)$$

Properties of $E_{r,\mathsf{M}}(\rho_{AB})$

- Asymptotic continuity

Proposition 3 Let ρ, ρ' be two states of dimension k, with $\|\rho - \rho'\|_{\mathsf{M}} \leq \epsilon \leq \frac{1}{e}$. Then

$$\left|E_{r,\mathsf{M}}(\rho) - E_{r,\mathsf{M}}(\rho')\right| \leq 2\epsilon \log \frac{6k}{\epsilon}.$$

- Evaluation on maximally entangled states and pure states

Proposition 4 For rank-d maximally entangled state Φ_d and pure state ψ_{AB},

$$E_{r,\mathsf{LO}}(\Phi_d) = E_{r,\text{1-LOCC}}(\Phi_d) = E_{r,\mathsf{LOCC}}(\Phi_d) = E_{r,\mathsf{SEP}}(\Phi_d) = E_{r,\mathsf{PPT}}(\Phi_d) = \log(d+1) - 1,$$

$$E^{\infty}_{r,\mathsf{LO}}(\psi_{AB}) = E^{\infty}_{r,\text{1-LOCC}}(\psi_{AB}) = E^{\infty}_{r,\mathsf{LOCC}}(\psi_{AB}) = E^{\infty}_{r,\mathsf{SEP}}(\psi_{AB}) = E^{\infty}_{r,\mathsf{PPT}}(\psi_{AB}) = S(\mathrm{Tr}_B\, \psi).$$

Comparisons between entanglement measures

We are mainly interested in two families of entanglement measures:

- Squashed-like measures $\{ E_{sq}, \quad E_I, \quad E_{sq,c} \}$
- E_r families $\{ E_{r,\rightarrow}, \quad E_{r,\leftrightarrow}, \quad E_r \}$

- Conditional entanglement of mutual information (Yang, Horodecki, Wang ' 08`

$$E_I(\rho_{AB}) := \tfrac{1}{2}\inf\{I(AA';BB')_\rho - I(A';B')_\rho\} \quad \text{with } \rho_{AA'BB'} \text{ being an extension of } \rho_{AB}$$

- C-squashed entanglement (Yang at el ' 07)

$$E_{sq,c}(\rho_{AB}) := \tfrac{1}{2}\inf\{I(A;B|E)_\rho\}, \text{ where extension state } \rho_{ABE} = \textstyle\sum_i p_i \rho^i_{AB} \otimes |\bar{i}\rangle\langle i|_E$$

- [illegible] ement (Pi[illegible])

$$E_{r,\rightarrow}(\rho_{AB}) := \sup_{\Lambda \in \text{LOCC}} E_{r,1\text{-LOCC}}(\Lambda(\rho_{AB})) \qquad E_{r,\leftrightarrow}(\rho_{AB}) := E_{r,\text{LOCC}}(\rho_{AB})$$

,

Comparisons between entanglement measures

(focusing mainly on regularized versions)

We obtain:

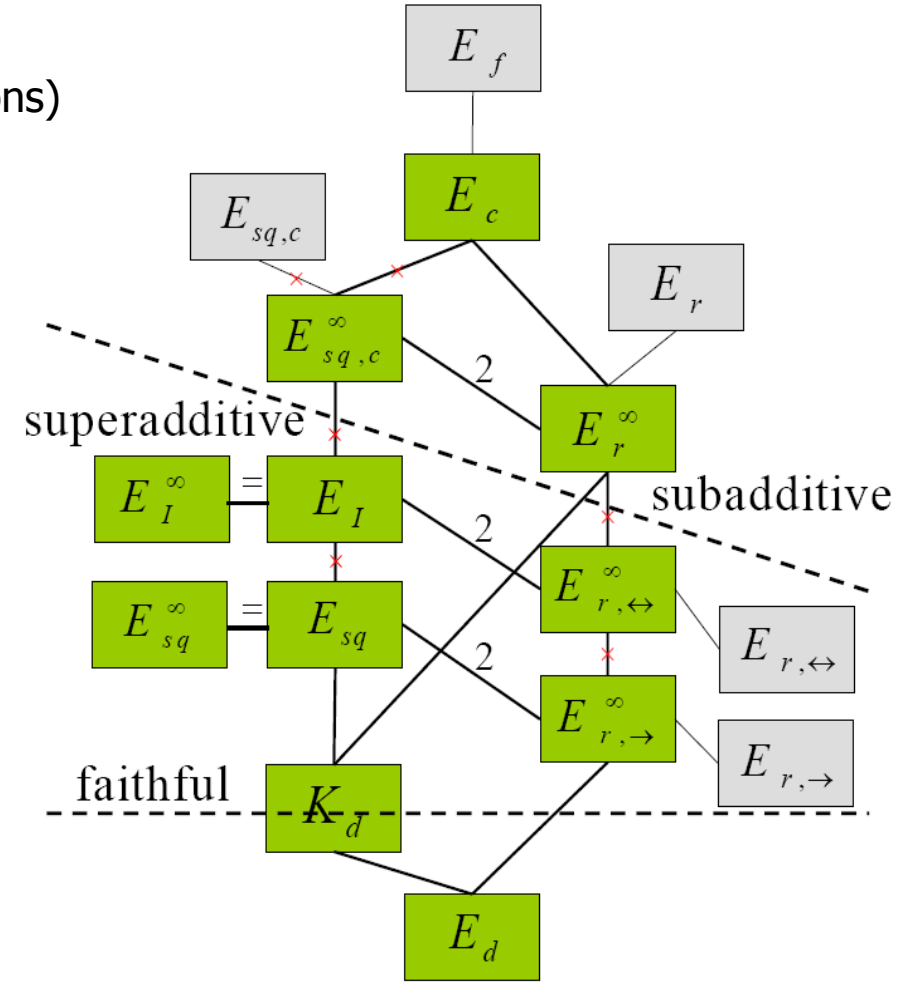
1. $2E^{\infty}_{sq,c} \geq E^{\infty}_{r}$,
2. $2E_I \geq E^{\infty}_{r,\leftrightarrow}$,
3. $2E_{sq} \geq E^{\infty}_{r,\rightarrow}$,
4. $E^{\infty}_{r,\rightarrow} \geq E_d$.

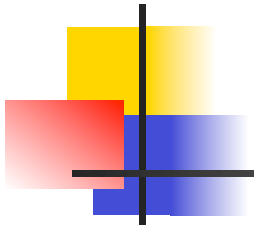
Previously known ones:

$$E_d \leq K_d \leq E_{sq} \leq E_I \leq E^{\infty}_{sq,c} \leq E_c,$$

$$E^{\infty}_{r,\rightarrow} \leq E^{\infty}_{r,\leftrightarrow} \leq E^{\infty}_{r},$$

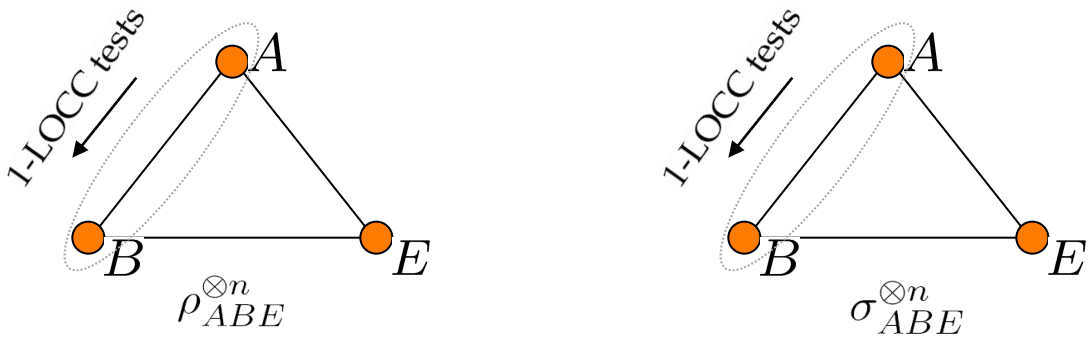
$$E_d \leq K_d \leq E^{\infty}_{r} \leq E_c.$$





Proofs of Theorem 1 and Theorem 2

Quantum hypothesis testing with one-way LOCC operations



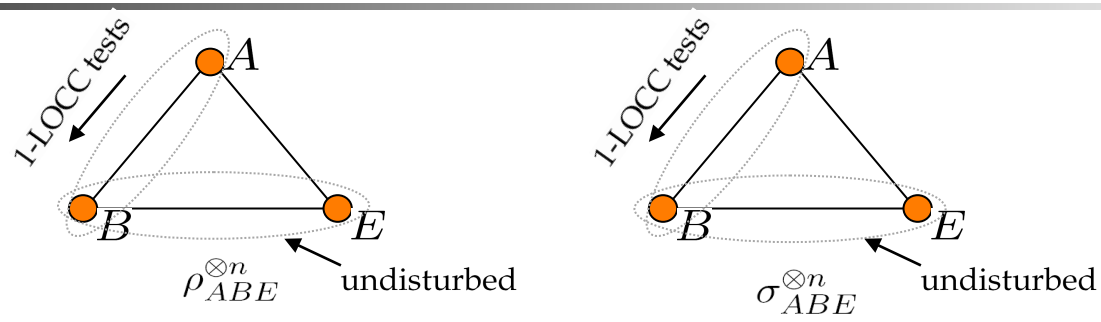
Consider the setting of hypothesis testing

two hypotheses: $\rho_{ABE}^{\otimes n}$ or $\sigma_{ABE}^{\otimes n}$

tests allowed: $\{L_n, \mathbb{1} - L_n\}$ on $A^n B^n$; 1-LOCC implementable

two errors: $\alpha_n(L_n) := \mathrm{Tr}(\rho^{\otimes n}(\mathbb{1} - L_n))$, $\beta_n(L_n) := \mathrm{Tr}(\sigma^{\otimes n} L_n)$

Quantum hypothesis testing with one-way LOCC operations



- For any 1-LOCC $\mathcal{M}^{AB\to X}$, there exist 1-LOCC tests $\{L_n, \mathbb{1} - L_n\}$ such that

  $\lim_{n\to\infty} \alpha_n(L_n) = 0,$

  $\lim_{n\to\infty} \frac{-1}{n} \log \beta_n(L_n) = D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma))$

  Lemma — Quantum Stein's (Hiai, Petz '91)

- Meanwhile, the states $\rho_{BE}^{\otimes n}$ and $\sigma_{BE}^{\otimes n}$ are kept almost undisturbed!

  Gentle Measurement Lemma (Winter '99)

Note: one-way LOCC measurement (1-LOC$\mathcal{M}^{AB\to XB}$ can be replaced by $\mathcal{M}^{AB\to X}$ one side local measurement (1-LM)

A technical lemma

Lemma *For any two states ρ_{ABE} and σ_{ABE}, and any one-way LOCC measurement $\mathcal{M}^{AB\to X}$ acting on system AB, with classical communication from A to B, there exists a sequence of quantum instruments $\mathcal{T}_n^{A^nB^n\to XB^n}$, which are implementable via local operations and classical communication from A^n to B^n, such that*

$$\lim_{n\to\infty}\frac{1}{n}D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n})\|\mathcal{T}_n^c(\sigma_{AB}^{\otimes n}))=D(\mathcal{M}(\rho_{AB})\|\mathcal{M}(\sigma_{AB})),$$
$$\lim_{n\to\infty}\left\|\mathcal{T}_n^q\otimes\mathbb{1}^{E^n}(\rho_{ABE}^{\otimes n})-\rho_{BE}^{\otimes n}\right\|_1=0,$$

where $\mathcal{T}_n^c:=\mathrm{Tr}_{B^n}\circ\mathcal{T}_n^{A^nB^n\to XB^n}$, and $\mathcal{T}_n^q:=\mathrm{Tr}_X\circ\mathcal{T}_n^{A^nB^n\to XB^n}$.

Note: one-way LOCC measurement (1-LOCC) $\mathcal{M}^{AB\to X}$ can be replaced by one side local measurement (1-LM) $\mathcal{M}^{AB\to XB}$

Proof of Theorem 1------
Monogamy relation for relative entropy of entanglement

To show $E_r(\rho_{B:AE}) \geq E_{r,\text{1-LOCC}}(\rho_{AB}) + E_r^{\infty}(\rho_{BE})$

Proof:

Let $\sigma_{B:AE}$ be the nearest separable state to $\rho_{B:AE}$;
Let $\mathcal{M}^{AB\to X}$ be an arbitrary one-way LOCC measurement;
Let $\mathcal{T}_n^{A^nB^n\to XB^n}$ be the quantum instruments in the lemma associated with ρ_{ABE}, σ_{ABE} and $\mathcal{M}^{AB\to X}$.

$$\begin{aligned} E_r(\rho_{B:AE}) &= D(\rho_{ABE}\|\sigma_{ABE}) \\ &= \frac{1}{n}D(\rho_{ABE}^{\otimes n}\|\sigma_{ABE}^{\otimes n}) \\ \text{(monotonicity)} \quad &\geq \frac{1}{n}D(\mathcal{T}_n\otimes\mathbb{1}^{E^n}(\rho_{ABE}^{\otimes n})\|\mathcal{T}_n\otimes\mathbb{1}^{E^n}(\sigma_{ABE}^{\otimes n})) \end{aligned}$$

Proof of Theorem 1------
Monogamy relation for relative entropy of entanglement

Write $\mathcal{T}_n \otimes \mathbb{1}^{E^n}(\rho_{ABE}^{\otimes n}) = \sum_{i_n} p_{i_n} |i_n\rangle\langle i_n|^X \otimes \rho_{B^n E^n}^{i_n}$

$\mathcal{T}_n \otimes \mathbb{1}^{E^n}(\sigma_{ABE}^{\otimes n}) = \sum_{i_n} q_{i_n} |i_n\rangle\langle i_n|^X \otimes \sigma_{B^n E^n}^{i_n}$

$$= \frac{1}{n} D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) + \frac{1}{n} \sum_{i_n} p_{i_n} D(\rho_{B^n E^n}^{i_n} \| \sigma_{B^n E^n}^{i_n})$$

(joint convexity)
$$\geq \frac{1}{n} D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) + \frac{1}{n} D(\mathcal{T}_n^q \otimes \mathbb{1}^{E^n}(\rho_{ABE}^{\otimes n}) \| \sum_{i_n} p_{i_n} \sigma_{B^n E^n}^{i_n})$$

(1-LOCC of Tn)
$$\geq \frac{1}{n} D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) + \frac{1}{n} E_r(\mathcal{T}_n^q \otimes \mathbb{1}^{E^n}(\rho_{ABE}^{\otimes n}))$$

(lemma & asym. cont.)
$$\stackrel{n\to\infty}{\longrightarrow} D(\mathcal{M}(\rho_{AB}) \| \mathcal{M}(\sigma_{AB})) + E_r^\infty(\rho_{BE}).$$

Since $\mathcal{M}$ is arbitrary, we have

$$E_r(\rho_{B:AE}) \geq \sup_{\mathcal{M} \in \text{1-LOCC}} D(\mathcal{M}(\rho_{AB}) \| \mathcal{M}(\sigma_{AB})) + E_r^\infty(\rho_{BE})$$
$$\geq E_{r,\text{1-LOCC}}(\rho_{AB}) + E_r^\infty(\rho_{BE}).$$

Proof of Theorem 2------

Commensurate lower bound for squashed entanglement

To show $$E_{sq}(\rho_{AB}) \geq \frac{1}{2} E^{\infty}_{r,1\text{-LOCC}}(\rho_{AB}) \geq \frac{1}{2} E_{r,1\text{-LOCC}}(\rho_{AB})$$

Proof:

(Theorem 1)
$$E^{\infty}_{r}(\rho_{B:AE}) \geq E^{\infty}_{r,1\text{-LOCC}}(\rho_{AB}) + E^{\infty}_{r}(\rho_{BE})$$

(Brandão, Christandl, Yard ’ 11, Lemma 1)
$$I(A;B|E)_{\rho} \geq E^{\infty}_{r}(\rho_{B:AE}) - E^{\infty}_{r}(\rho_{BE})$$

$$\Longrightarrow E_{sq}(\rho_{AB}) \geq \frac{1}{2} E^{\infty}_{r,1\text{-LOCC}}(\rho_{AB}) \geq \frac{1}{2} E_{r,1\text{-LOCC}}(\rho_{AB})$$

Last remark

$$E_r(\rho_{B:AE}) \geq E_{r,\text{1-LOCC}}(\rho_{AB}) + E_r^{\infty}(\rho_{BE})$$

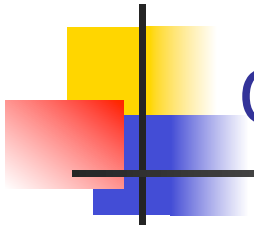
$$E_r^{\infty}(\rho_{B:AE}) \geq E_{r,\text{1-LOCC}}^{\infty}(\rho_{AB}) + E_r^{\infty}(\rho_{BE})$$

$$E_{sq}(\rho_{AB}) \geq \frac{1}{2}E_{r,\text{1-LOCC}}^{\infty}(\rho_{AB}) \geq \frac{1}{2}E_{r,\text{1-LOCC}}(\rho_{AB})$$

1 - LM

One-way LOCC measurement can be replaced by one-side local measurement, which is a measurement on system A and an identity operation on system B.

Inspired by the results of (Brandão & Harrow ’ 12)



Open questions

- Applications of our results?
- Faithfulness of multipartite squashed entanglement?



Thank you!