ON THE ADDITIVE AND MULTIPLICATIVE ADVERSARY METHODS

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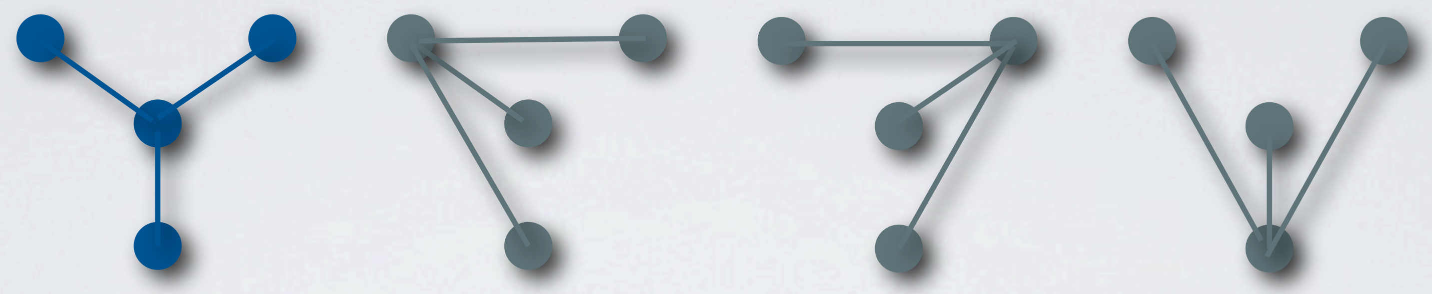
QIP'11, Singapore, 1.11.11

QUANTUM STATE GENERATION

[Folklore] One way to solve Graph Isomorphism

Create the uniform superposition on permuted graphs



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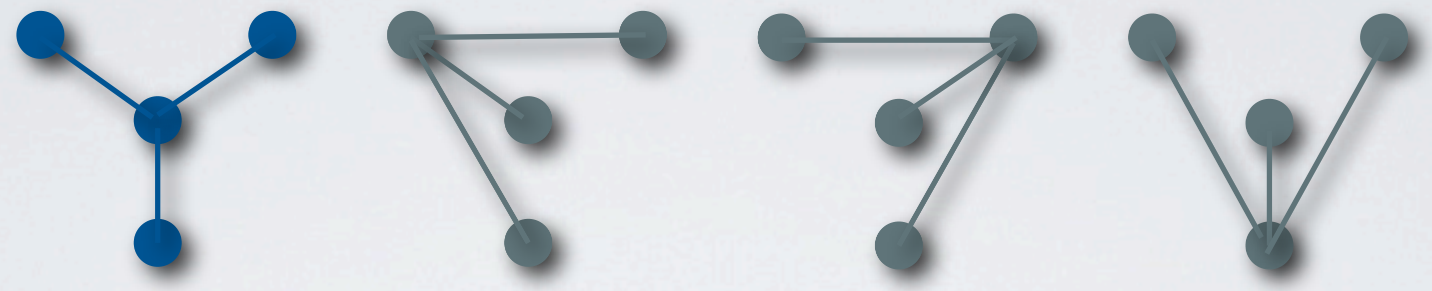

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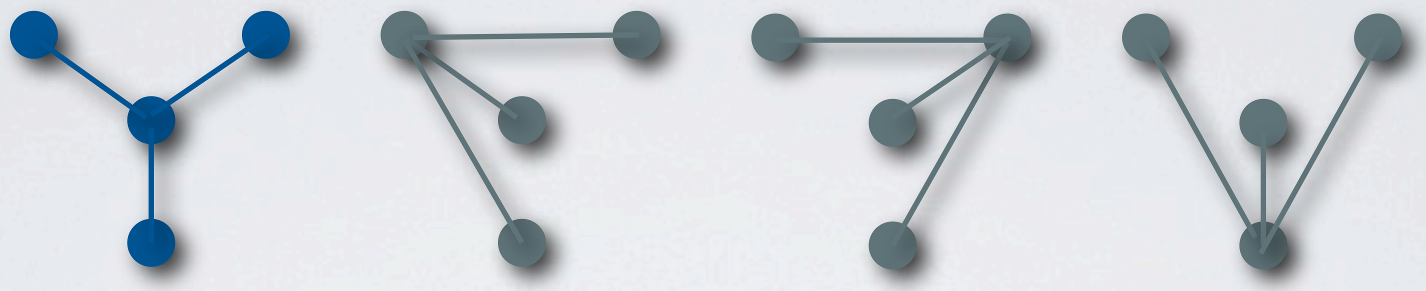

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- Let's try something simpler: query complexity

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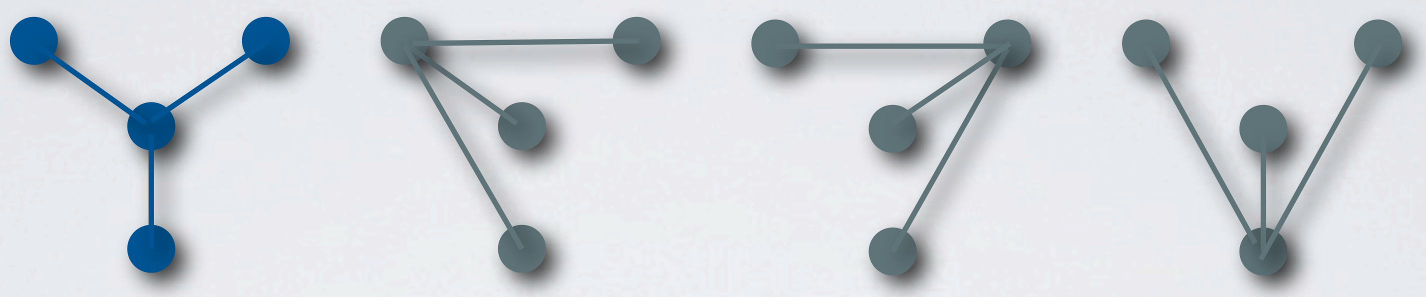

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f injective $[N] \to [M]$

$$\frac{1}{\sqrt{N}} \sum_{x \in [N]} |f(x)\rangle \qquad\qquad \frac{1}{\sqrt{N}} \sum_{x \in [N]} |x\rangle |f(x)\rangle$$

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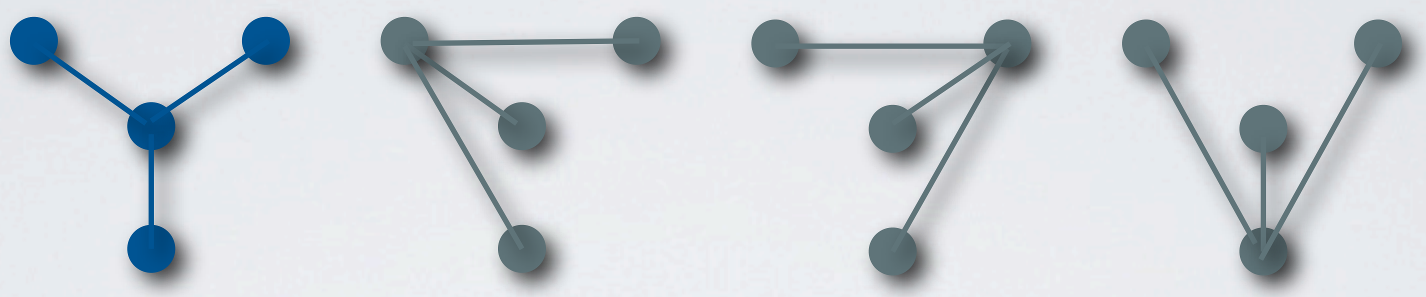

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INDEX ERASURE PROBLEM [Shi'02]

Given an injective function $f : [N] \longrightarrow [M]$ as an oracle
create the state $|\psi_f^{\odot}\rangle = \frac{1}{\sqrt{N}} \sum_{x \in [N]} |f(x)\rangle$

QUERY COMPLEXITY

QUANTUM STATE GENERATION PROBLEM

Given a function $f \in F$ as an oracle $O_f : |x\rangle|s\rangle \mapsto |x\rangle|f(x) \oplus s\rangle$ create a state ε-close to a target state $|\psi_f^{\odot}\rangle$

$Q_\varepsilon(\psi)$ Minimal number of queries that solve the problem over all algorithms.

QUANTUM PROBLEMS

Example: Index Erasure

CLASSICAL PROBLEMS (Tutorial by Ben Reichardt)

Creating $|\psi_f\rangle$ is computing a function

Example: search $\psi_f = \bigoplus_i f(x_i)$

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NEW

joint work with Ambainis

Complexity $\Theta(\sqrt{N})$

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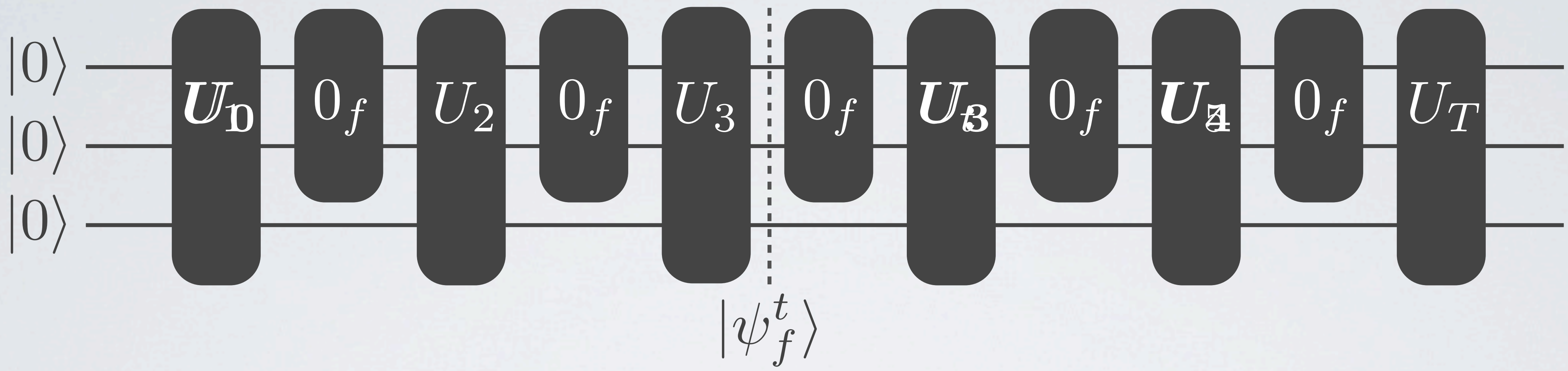

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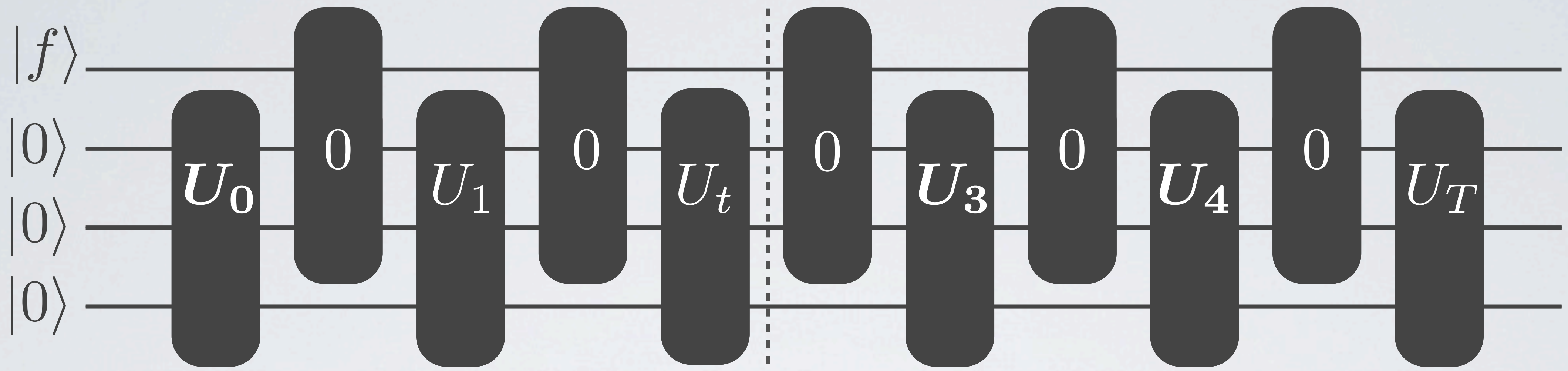
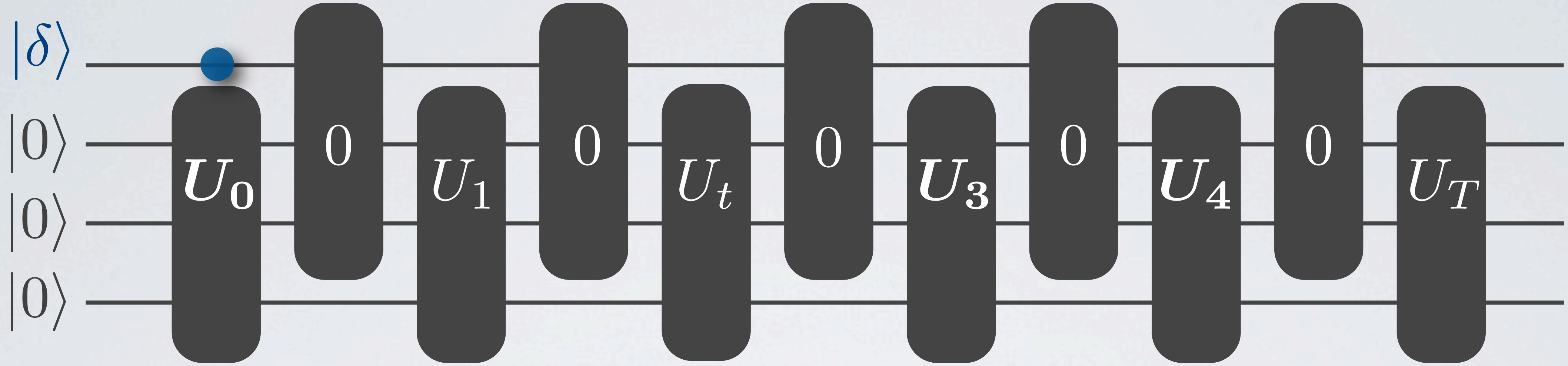


$$|\Psi_f^t\rangle = |f\rangle|\psi_f^t\rangle$$

PRINCIPLES BEHIND THE ADVERSARY METHODS

$|\delta\rangle = 1/|F| \sum |f\rangle$



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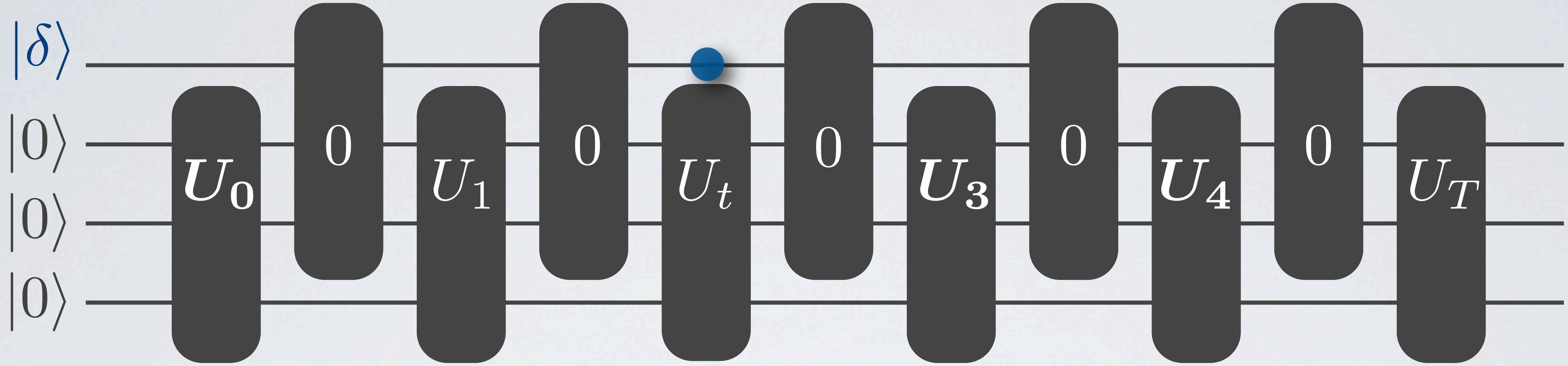

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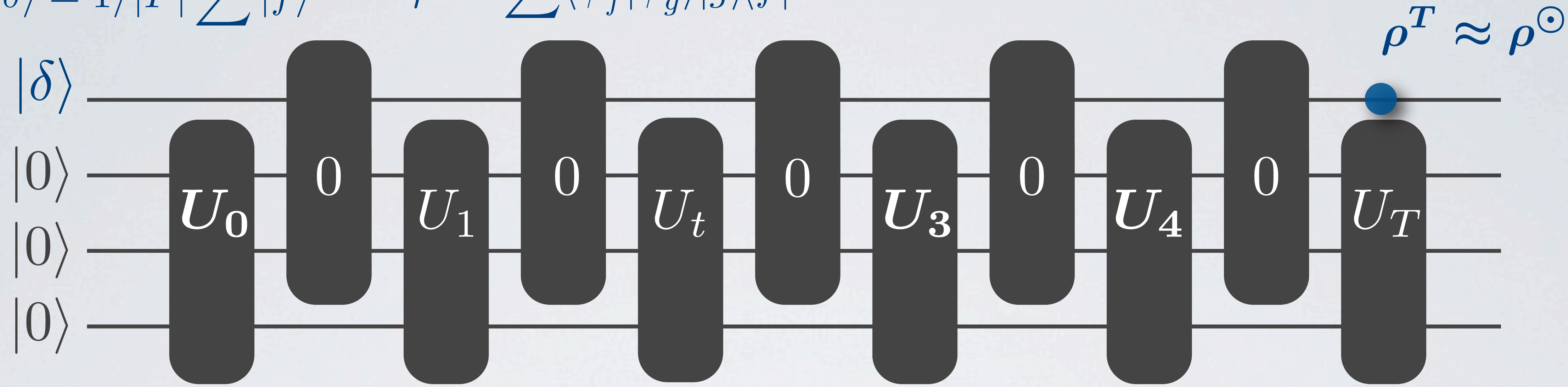

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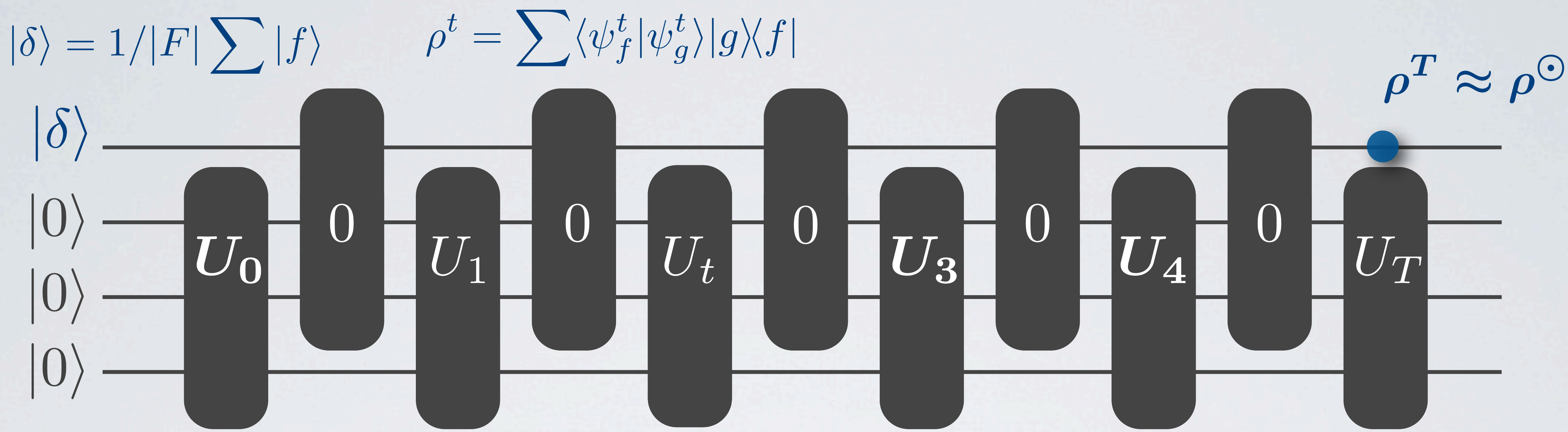
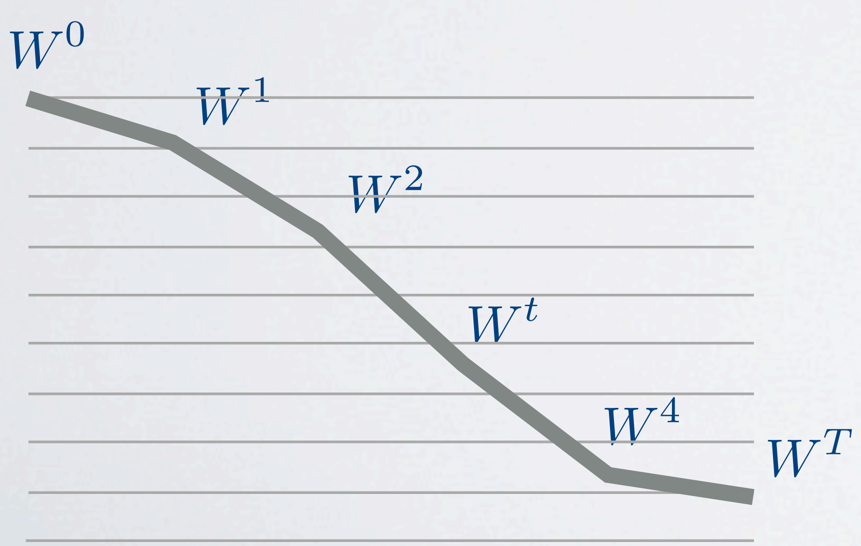


Progress function $\boldsymbol{W^t = \sum \Gamma_{fg} \langle \psi_f^t | \psi_g^t \rangle = \mathrm{tr}[\Gamma \rho^t]}$



- Initial value (high)
- Progress done by one query (limited)
- Final value (low, depends on the success probability)

ADVERSARIES FOR CLASSICAL PROBLEMS

97 [BBBV'97] Hybrid argument

00 [Amb'00] Adversary method

03 [Amb'03]

04 [LM'04] Several variations (and many more)

05 [Amb'05]

06 [SŠ'06] All these methods are equivalent

07 [HLŠ'07] Additive method (negative weights) $W^t - W^{t+1}$

08 [Špa'08] Multiplicative method W^t / W^{t+1}

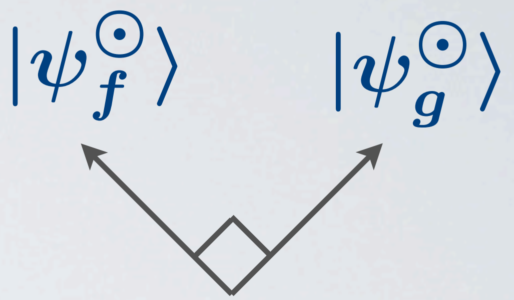
09 [Reï'09] The additive method is tight for all functions

10 [LMRŠ'10] in the bounded-error model

ADVERSARY MATRICES, FIRST APPROACH

$$W^t = \mathrm{tr}[\Gamma \rho^t] = \sum \Gamma_{fg} \langle \psi_f^t | \psi_g^t \rangle$$

For computing functions (classical) :



Conditions on Γ: ❶ definite positive

❷ $\Gamma_{fg} = 0$ when $|\psi_f^{\odot}\rangle = |\psi_g^{\odot}\rangle$

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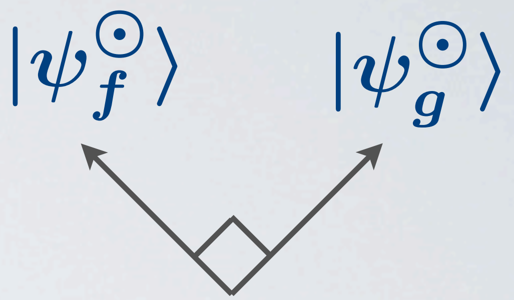

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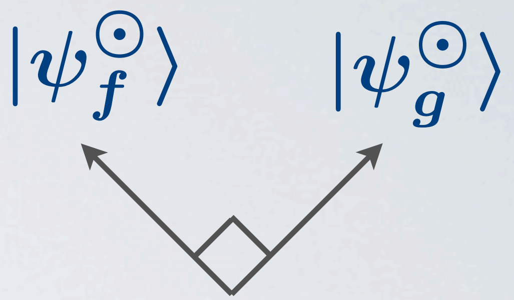

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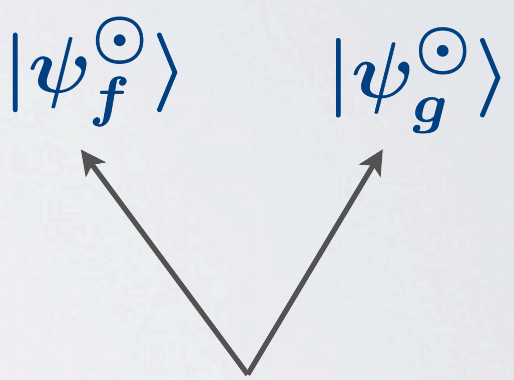
[this work]

For quantum state generation:

- non-orthogonal output states



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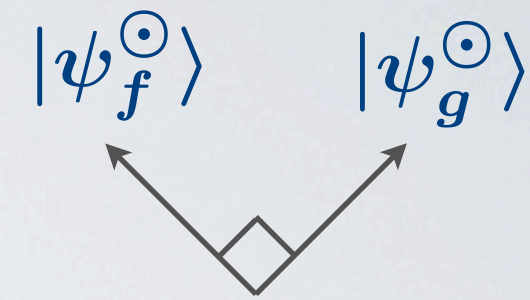

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too restrictive for small success probability

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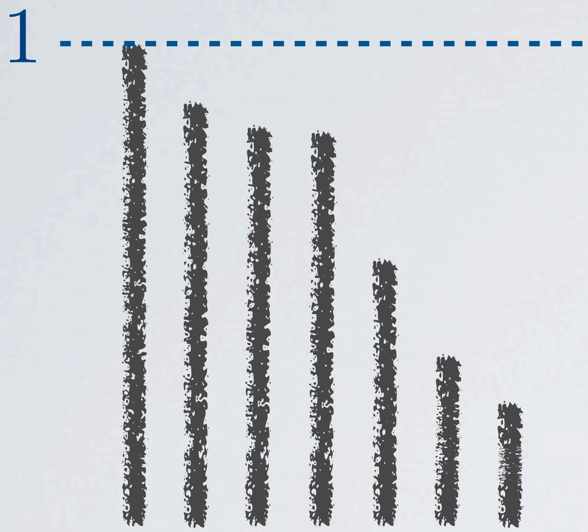
NEW METHOD: HYBRID

Eigenvalues of $\mathbf{\Gamma}$



Conditions on $\mathbf{\Gamma}$, hybrid method

$\mathbf{\Gamma} \preceq \mathbb{I}$

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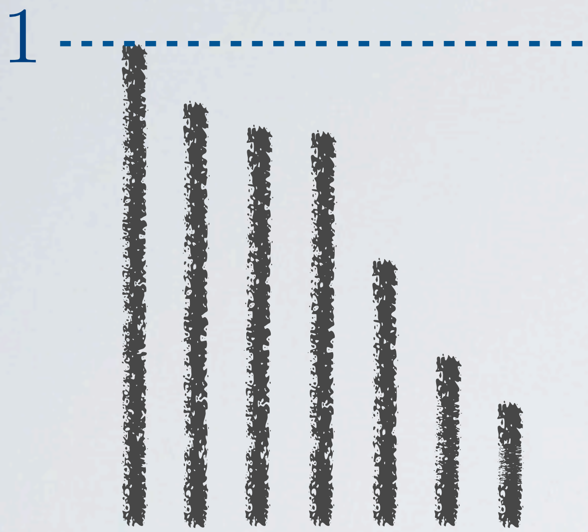

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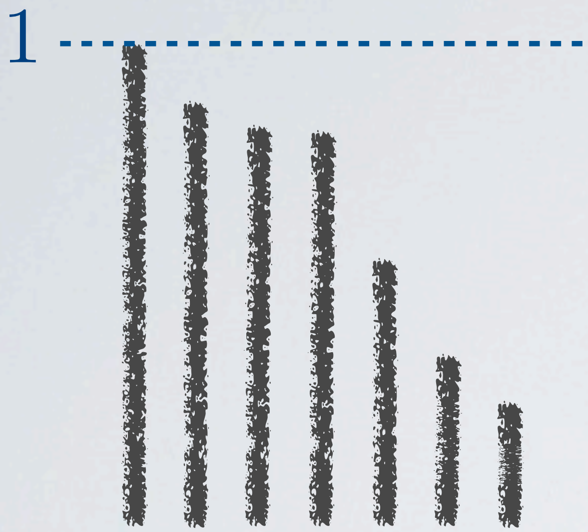

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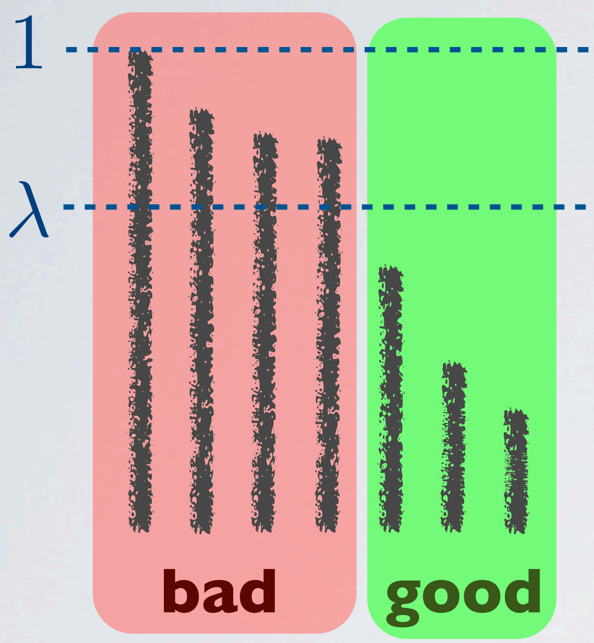

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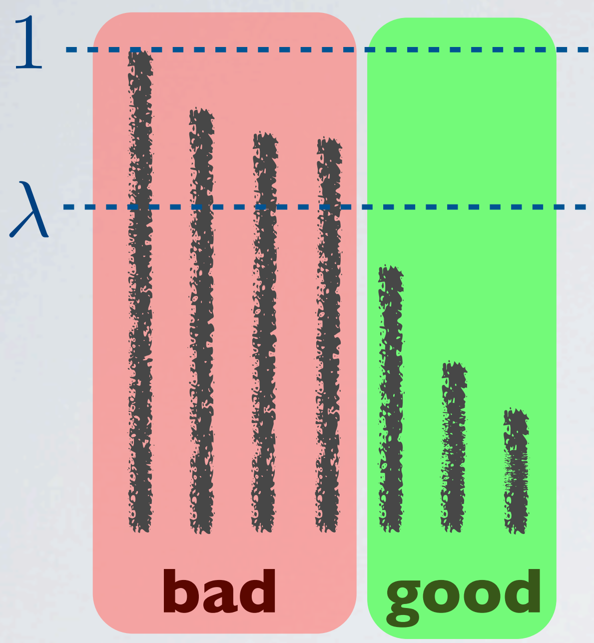

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Initial value: $W^0 = 1$

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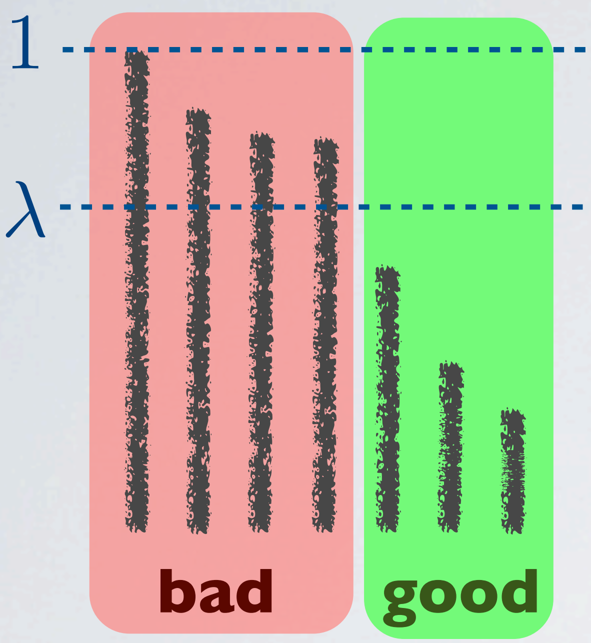

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THEOREM

$\mathrm{MADV}_\varepsilon \geq \mathrm{ADV}_\varepsilon^{\mathrm{Hyb}} \geq \mathrm{ADV}_\varepsilon^{\pm} / 60$

SYMMETRIZATION

2 technical difficulties:

- Designing a « good » adversary matrix
- Computing the norm $||\boldsymbol{\Gamma}_{\boldsymbol{x}} - \boldsymbol{\Gamma}||$

Solution:

- Using the symmetries of the problem

Index Erasure: $|\psi_{f}^{\odot}\rangle = \frac{1}{\sqrt{N}} \sum_{x \in [N]} |f(x)\rangle$ $\quad f : [N] \mapsto [M]$

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The circuit should have this symmetry

USING SYMMETRIES

- π permutation on the inputs
- τ permutation on the outputs

$$f_{\pi,\tau} = \tau \circ f \circ \pi$$

$$U_{\pi\tau}|f\rangle = |f_{\pi\tau}\rangle$$

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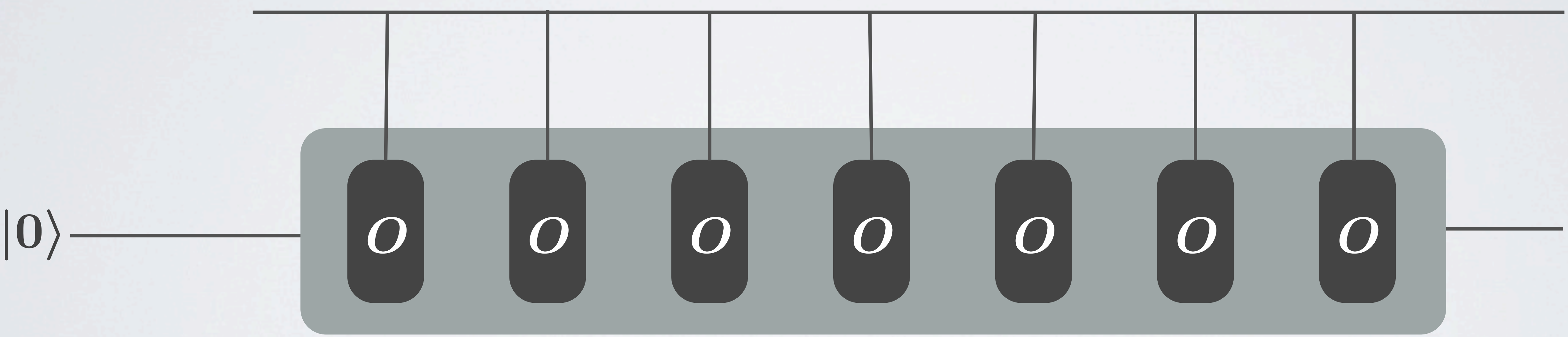

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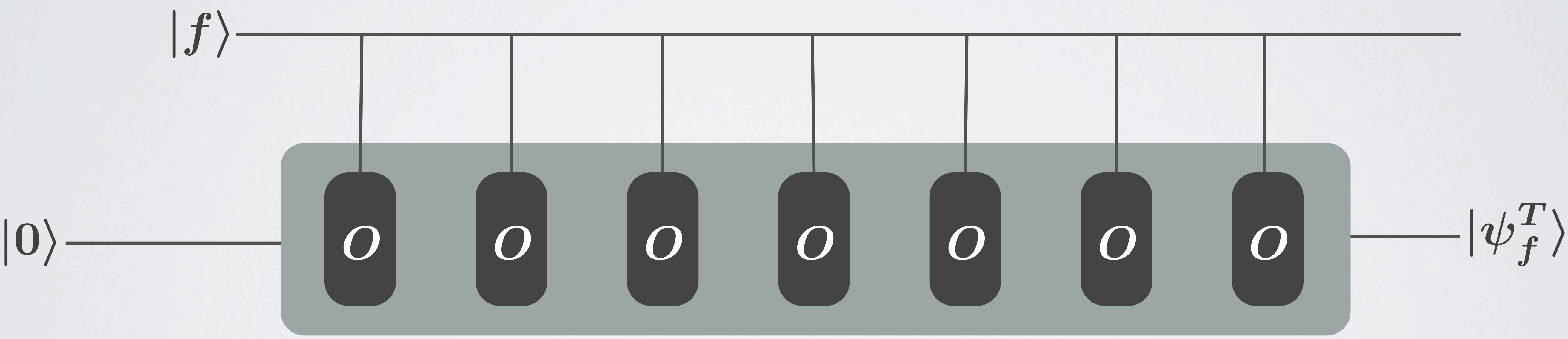

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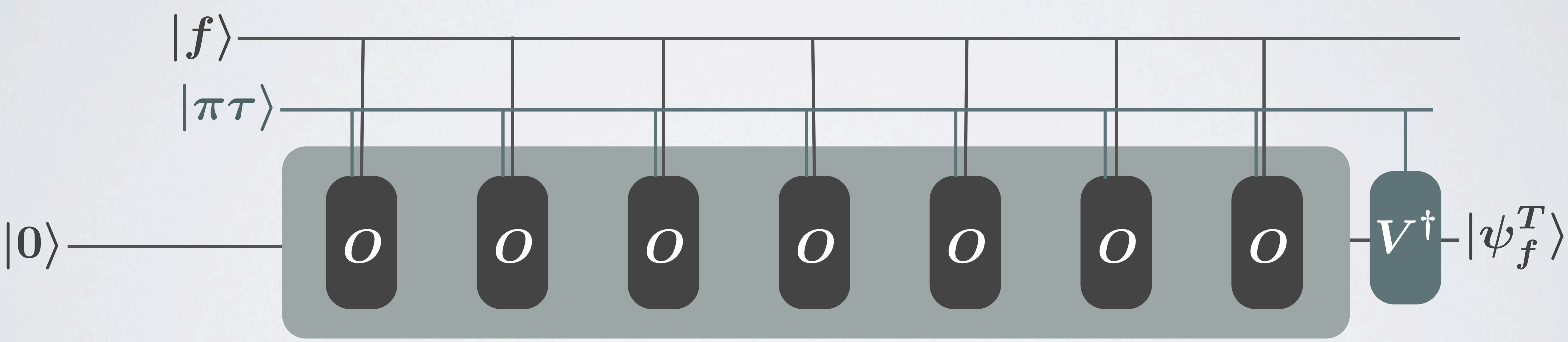

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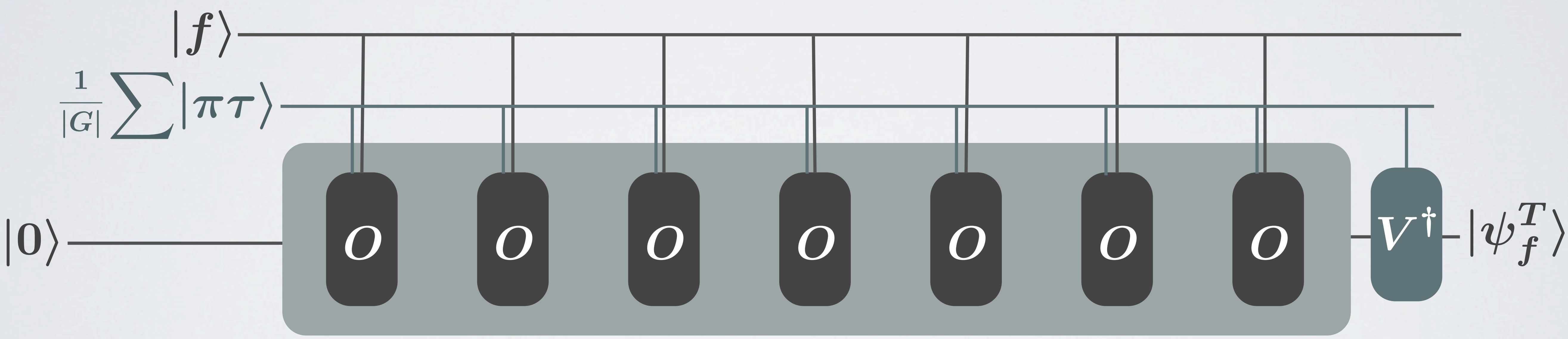

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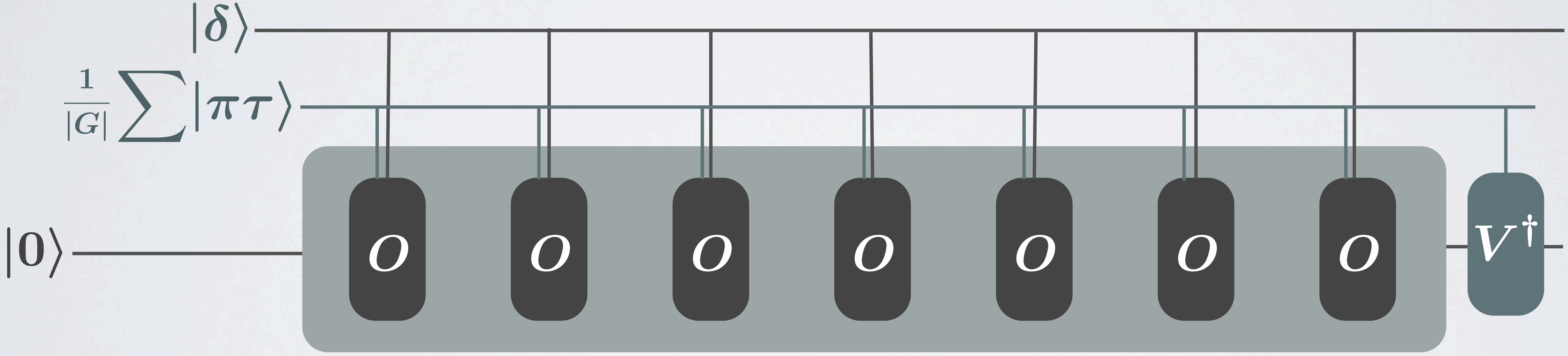

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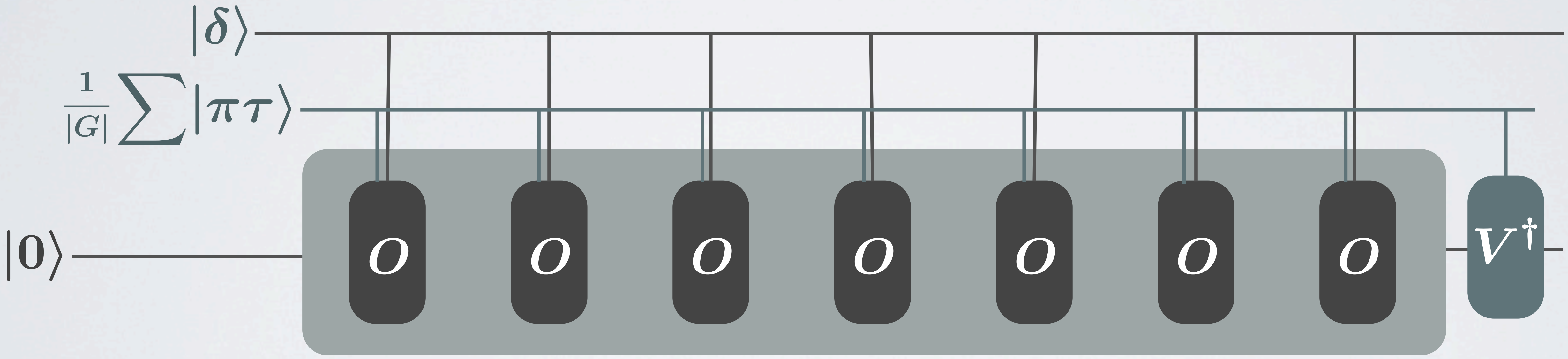

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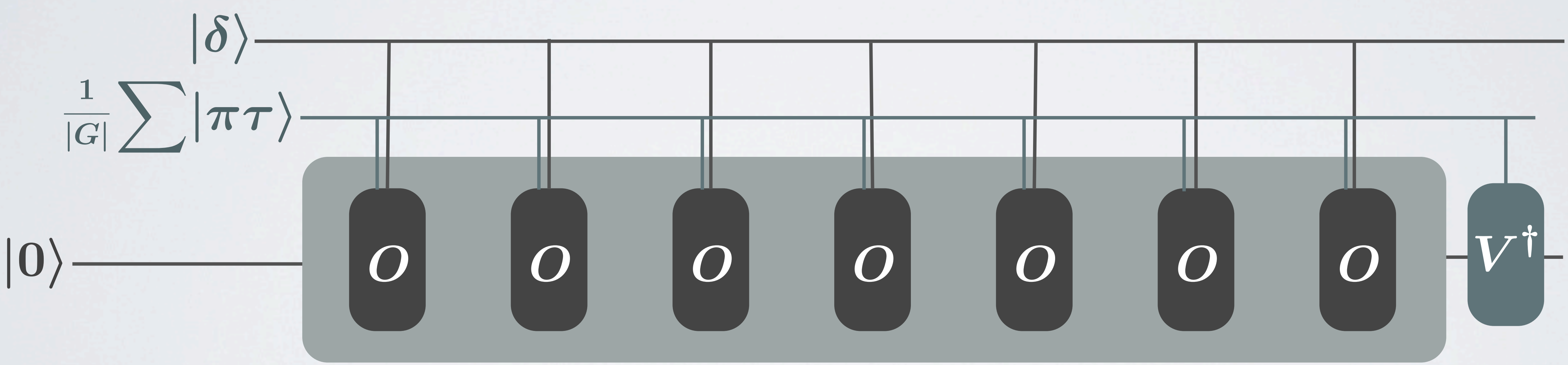

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$\mathcal{U} : (\pi,\tau) \mapsto U_{\pi\tau}$ is a representation of G

USING SYMMETRIES WHEN $\mathcal{U}$ IS MULTIPLICITY-FREE

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$$G_x = \{(\pi, \tau) \in G \;:\; \pi(x) = x\}$$

$$G_{xy} = \{(\pi, \tau) \in G \;:\; \pi(x) = x, \tau(y) = y\}$$

l is an irrep of G_x with multiplicity m_l

$$\|\boldsymbol{\Gamma}_x - \boldsymbol{\Gamma}\| = \max_l \left\|\boldsymbol{\Delta}_x^l\right\|$$

USING SYMMETRIES WHEN $\mathcal{U}$ IS MULTIPLICITY-FREE

$\boldsymbol{\Gamma = \sum_k \gamma_k \Pi_k}$ where $\boldsymbol{\Pi_k}$ is the $\boldsymbol{k}$-th irrep of $\boldsymbol{G}$

$$\boldsymbol{G_x = \{(\pi,\tau) \in G \ : \ \pi(x) = x\}}$$

$$\boldsymbol{G_{xy} = \{(\pi,\tau) \in G \ : \ \pi(x) = x, \tau(y) = y\}}$$

depends on the overlap of irreps of G, G_x and G_{xy}

$\boldsymbol{l}$ is an irrep of $\boldsymbol{G_x}$ with multiplicity $\boldsymbol{m_l}$

$$\boldsymbol{\|\Gamma_x - \Gamma\| = \max_l \left\|\Delta_x^l\right\|}$$

large size: |F| x |F|
index erasure $N!\binom{M}{N}$.

small size: $m_l \times m_l$
index erasure 3×3

USING SYMMETRIES WHEN $\mathcal{U}$ IS MULTIPLICITY-FREE

$$\boldsymbol{\Gamma} = \sum_k \gamma_k \boldsymbol{\Pi}_k$$ where $\boldsymbol{\Pi}_k$ is the k-th irrep of G

$$G_x = \{(\pi, \tau) \in G \;:\; \pi(x) = x\}$$

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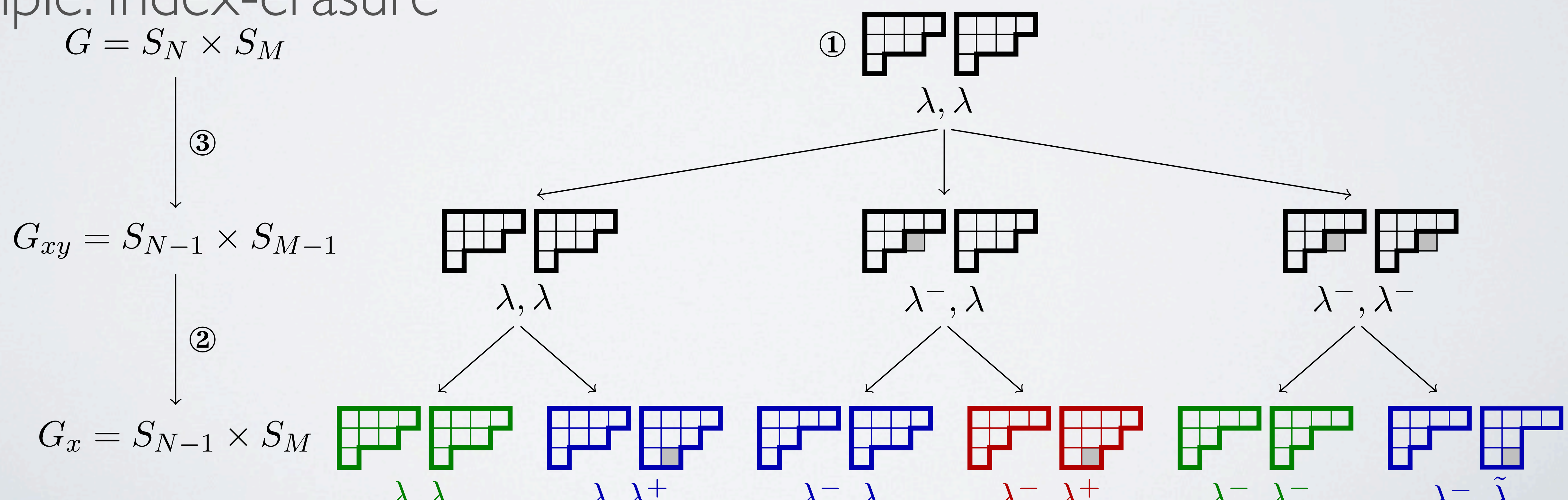
l is an irrep of G_x with multiplicity m_l

depends on the overlap of irreps of G, G_x and G_{xy}

$$\|\boldsymbol{\Gamma}_x - \boldsymbol{\Gamma}\| = \max_l \left\| \boldsymbol{\Delta}_x^l \right\|$$

Example: Index-erasure



STRONG DIRECT PRODUCT THEOREM

Assume: multiplicative lower bound complexity for QSG is T.
Complexity of solving QSG on k independent instances?

- Upper bound $O(kT)$
- Lower bound?

STRONG DIRECT PRODUCT THEOREM

Assume: multiplicative lower bound complexity for QSG is T.
Complexity of solving QSG on k independent instances?

- Upper bound $O(kT)$
- Lower bound?

STRONG DIRECT PRODUCT THEOREM

The success probability of an algorithm solving QSG on k independent instances with less than $kT/10$ queries is exponentially small in k.

$$\mathrm{MADV}_{\epsilon'}^{(k)} \leq \frac{k}{10}\mathrm{MADV}_{\epsilon}$$

OPEN QUESTIONS

QUERY COMPLEXITY

- Optimality for quantum state generation?
- Strong direct product theorem holds for *all functions*?
  - quantum state generation problems?

PROVING LOWER BOUNDS

- New lower bounds? (Set equality)
- Shorter/Simpler proofs?
- What about Graph Isomorphism?

Acknowledgments of support:

Thank you for your attention!

Methods + Application to Index erasure:
arXiv:1012.2112 [quant-ph]

A SHORT BIBLIOGRAPHY

[Amb'00]	STOC'00
[Amb'03]	FOCS'03
[Amb'05]	arxiv:quant-ph/0508200
[BBBV'97]	SIAM J. Comput, 1997
[HLŠ'07]	STOC'07
[LM'04]	CCC'04
[LMRŠ'10]	arxiv:1011.3020 [quant-ph]
[Rei'09]	FOCS'09
[Špa'08]	CCC'08
[Shi'02]	FOCS'02
[SŠ'06]	Th. Comput. 2006