Finding is as easy as detecting for quantum walks

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NEC

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[ICALP'2010, arxiv:1002.2419]

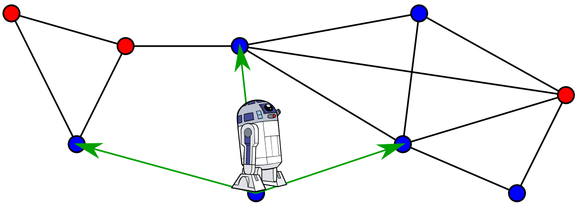
Spatial search on a graph

Setup

- Graph G on n vertices X
- Marked vertices: unknown $M \subseteq X$
- Vertex register: “robot” position
- Edges: legal moves

The problem

- Move the robot to a marked vertex $x \in M$
- Complexity: # moves



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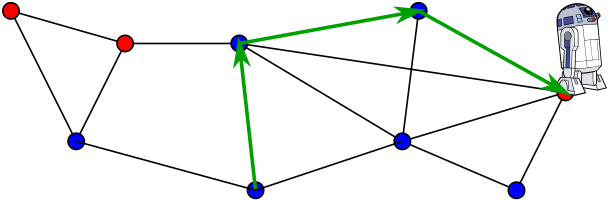


Search via random walk

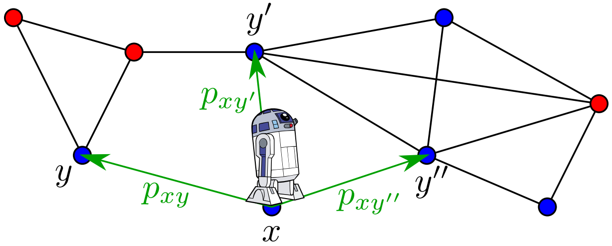
Markov chain on the graph

Stochastic matrix $P = (p_{xy})$

- $p_{xy} \neq 0$ only if (x, y) is an edge
- Stationary distribution π $(\pi P = \pi)$

Algorithm

- Start from random $x \sim \pi$
- Apply P until x is marked



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Expected # steps of P until $x \in M$

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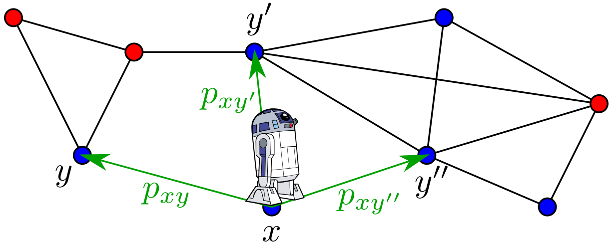

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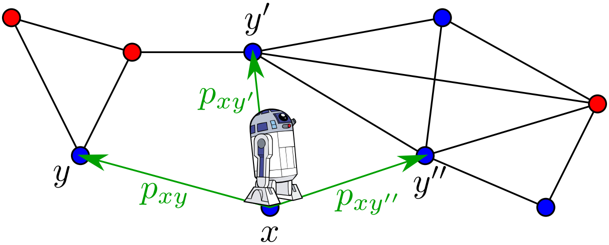

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Quantum case: Related work

- Quantum walks
  - Complete graph [Grover'95]
  - Hypercube [Shenvi,Kempe,Wayley'03]
  - Johnson Graph [Ambainis'04]
  - 2D-grid [Ambainis,Kempe,Rivosh'05]
  - Quantum analogue $W(P)$ of Markov chain P [Szegedy'04]

- Quantum hitting time
  - Detecting marked elements: $\sqrt{\mathrm{HT}(P,M)}$ [Szegedy'04]
  - Finding marked elements for state-transitive P and $|M|=1$: $\sqrt{\mathrm{HT}(P,M)}$ [Tulsi'08][Magniez,Nayak,Richter,Santha'09]

Question

Is finding as easy as detecting for quantum walks?

$$\mathrm{QHT}(P,M) \stackrel{?}{=} \sqrt{\mathrm{HT}(P,M)}$$

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Algorithmic applications

- Grover Search [Grover'95]
  - Search for a 1 in an n-bit string
  - G: complete graph
  - Classical: n Quantum: $\sqrt{n}$
  - Extends to G hypercube and unique marked element ($|M| = 1$)
- Element Distinctness [Ambainis'04]
  - Search for equal elements in a set of n elements
  - G: Johnson graph
  - Classical: n Quantum: $n^{2/3}$
- Triangle Finding [Magniez,Santha,Szegedy'05]
  - Search for a triangle in a graph with n vertices
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- Others
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Our main result

Theorem

Let

- P be a reversible, ergodic Markov chain
- π be the (unique) stationary distribution of P
- $\epsilon = \Pr_\pi(M)$ be the probability of marked elements

Then, there exists a quantum algorithm that finds an element in M within

- $\sqrt{\mathrm{HT}(P,M)}$ steps if ϵ is known
- $\sqrt{\mathrm{HT}(P,M) \times \log n}$ steps otherwise

Quadratic speed-up for any reversible P!

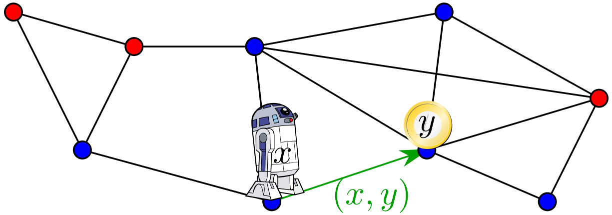
From random to quantum walks [Szegedy'04]

Random walk P on edges (x, y)

- Acts on two registers: position x and coin y
- Walk in two steps:
  - Flip the coin y over the neighbours of x
  - Swap x and y

Quantum analogue $W(P)$

- Acts on two registers $|x\rangle|y\rangle$
- Walk in two steps:
  - reflection of $|y\rangle$ through $|p_x\rangle = \sum_{y'} \sqrt{p_{y'x}}|y'\rangle$
  - Swap the $|x\rangle$ and $|y\rangle$ registers



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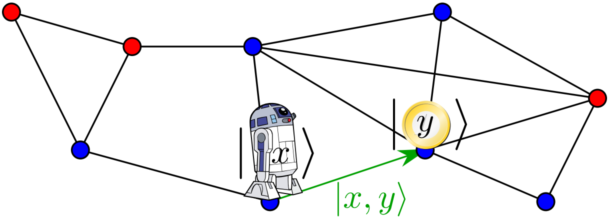
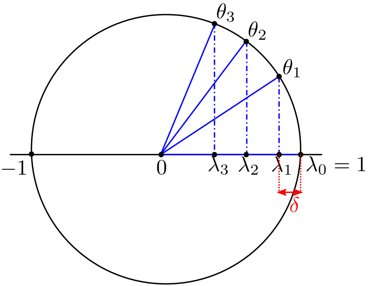


Random walk

- $P = (p_{xy})$
- E-v: $\lambda_k = \cos\theta_k$
- Stationary dist. ($\cos\theta_0 = 1$):
  $$\pi = (\pi_x)$$
- E-v gap: $\delta = 1 - |\cos\theta_1|$

Quantum walk

- $W(P) = \textsc{Swap} \cdot \mathrm{ref}_{\mathcal{X}}$
- E-v: $e^{\pm i\theta_k}$
- Stationary state ($\theta_0 = 0$):
  $$|\pi\rangle = \sum_x \sqrt{\pi_x}|x\rangle|p_x\rangle$$
- phase gap: $\Delta = |\theta_1| = \Theta(\sqrt{\delta})$



Spectral correspondance [Szegedy'04]

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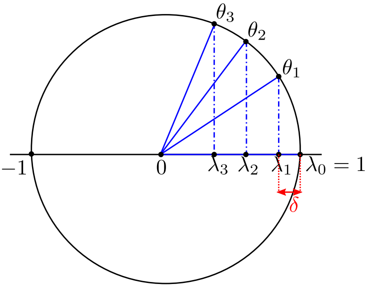

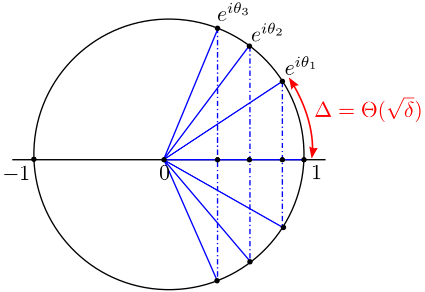

Absorbing walk

Recall:

- Reversible, ergodic Markov chain P
- (unique) stationary distribution π
- Set of marked elements M:

$$P = \begin{pmatrix} P_{UU} & P_{UM} \\ P_{MU} & P_{MM} \end{pmatrix}$$

Absorbing walk P'

- Same as P but self-loops for marked vertices
- Stationary distribution π_M: π restricted to marked vertices
- Hitting time $\mathrm{HT}(P,M) = \sum_{\lambda'_k \neq 1} \frac{|\langle v'_k|\pi\rangle|^2}{1-\lambda'_k}$ = "# steps of P' to map $\pi \mapsto \pi_M$"

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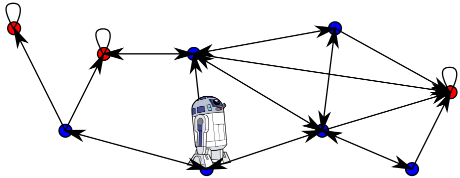

Quantum analogues of P and P'

Absorbing walk P'

- $\sqrt{\mathrm{HT}(P,M)}$ iterations of $W(P')$ make $|\pi\rangle$ deviate by angle $\Omega(1)$
  - Good for detecting if M is non-empty [Szegedy'04]
- But: state may remain far from marked elements
  - Can be fixed for state-transitive P, $|M| = 1$
  - Difficult analysis, less intuition [Tulsi'08][Magniez,Nayak,Richter,Santha'09]

Original walk P

- Extends Grover's algorithm for any graph
  - Good for finding [Ambainis'04][Magniez,Nayak,Roland,Santha'07]
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New approach: mixture of P and P'

- Finds marked elements for any reversible P, and any $|M|$
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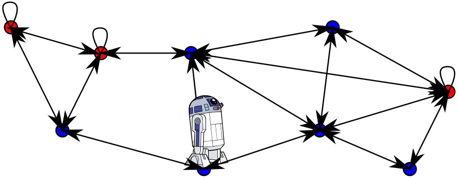
Interpolation between P and P'

- $P(s) = (1-s)P + sP'$
  - Unmarked vertices: apply P
  - Marked vertices: apply P with probability $1-s$, otherwise self-loop



- Stationary distribution $\pi(s) = (\cos^2\phi(s))\pi_U + (\sin^2\phi(s))\pi_M$
  - where $\phi(s) = \arcsin\sqrt{\frac{\epsilon}{1-s(1-\epsilon)}}$
  - Similarly, $|\pi(s)\rangle = \cos\phi(s)|\pi_U\rangle + \sin\phi(s)|\pi_M\rangle$
  - Rotates from $|\pi\rangle = \sqrt{1-\epsilon}|\pi_U\rangle + \sqrt{\epsilon}|\pi_M\rangle$ to $|\pi_M\rangle$
- Reminiscent of adiabatic quantum computing
  - Indeed, we can also design an adiabatic algorithm [Krovi,Ozols,R.'10, PRA]
  - Note: Interpolation at the classical level

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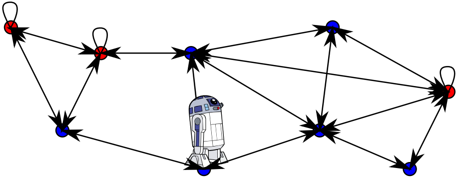

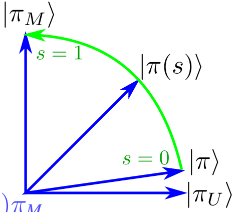

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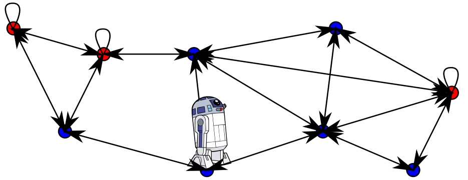

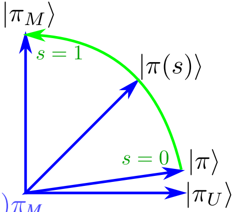

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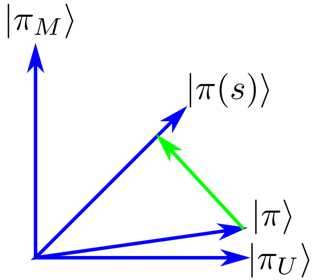
The algorithm

General idea

- Using quantum phase estimation [Kitaev'95][Cleve,Ekert,Macchiavello,Mosca'98]
  - We can measure in the eigenbasis of $W(P(s))$
  - At a cost $\sqrt{\mathrm{HT}(s)}$ (see later)
- $W(P(s))$ has unique 1-eigenvector $|\pi(s)\rangle$
  - Measuring phase 0 projects onto $|\pi(s)\rangle$

Algorithm (known ϵ)

- Prepare $|\pi\rangle$
- Project onto $|\pi(s^*)\rangle = \frac{1}{\sqrt{2}}(|\pi_U\rangle + |\pi_M\rangle)$
  - succeeds with prob. $\approx 1/2$
- Measure current vertex
  - marked with prob. $1/2$



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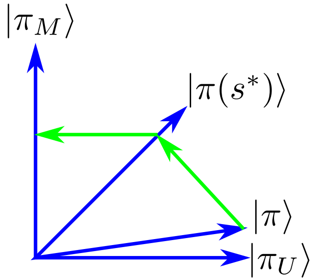
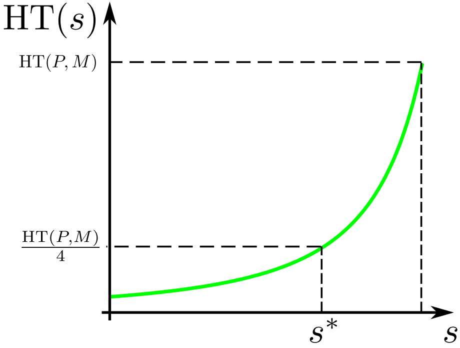


Interpolated hitting time

- "Interpolated hitting time"
  $\mathrm{HT}(s) = \sum_{\lambda_k(s)\neq 1} \frac{|\langle v_k(s)|\pi\rangle|^2}{1-\lambda_k(s)}$ ="# steps of $P(s)$ to map $\pi \mapsto \pi(s)$"
- We show: $\mathrm{HT}(s) = \sin^4\phi(s)\cdot \mathrm{HT}(P,M)$



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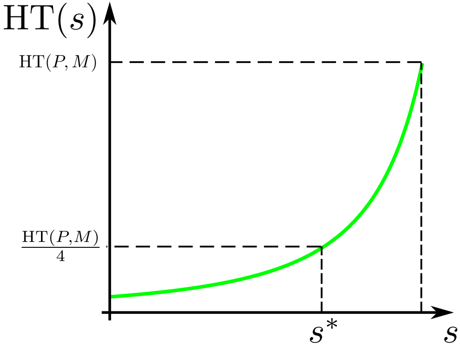

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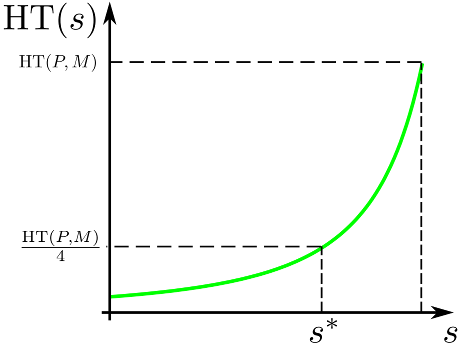

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Conclusion

Our contribution

- There exists a quantum algorithm that finds an element in M within
  - $\sqrt{\mathrm{HT}(P,M)}$ steps, if ϵ is known
  - $\sqrt{\mathrm{HT}(P,M) \times \log n}$ steps, otherwise
- Application: 2D-grid, finding an element within
  - $\sqrt{n \log n}$ steps, if ϵ is known
  - $\sqrt{n} \log n$ steps, otherwise

Open problems

- Hitting time
  - Can we beat the quadratic improvement?
- Mixing time
  - Can we also mix quadratically faster using quantum walks?
  - Very few results for Cayley graphs [Aharonov,Ambainis,Kempe,Vazirani'01]

Support:

Conclusion

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