The Fidelity Alternative and Quantum Measurement Simulation

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Andreas Winter (Bristol, Singapore)

arXiv: out.soon (2009) ...

Outline

1. What does it mean to transmit a state?

2. Simulation of binary measurements

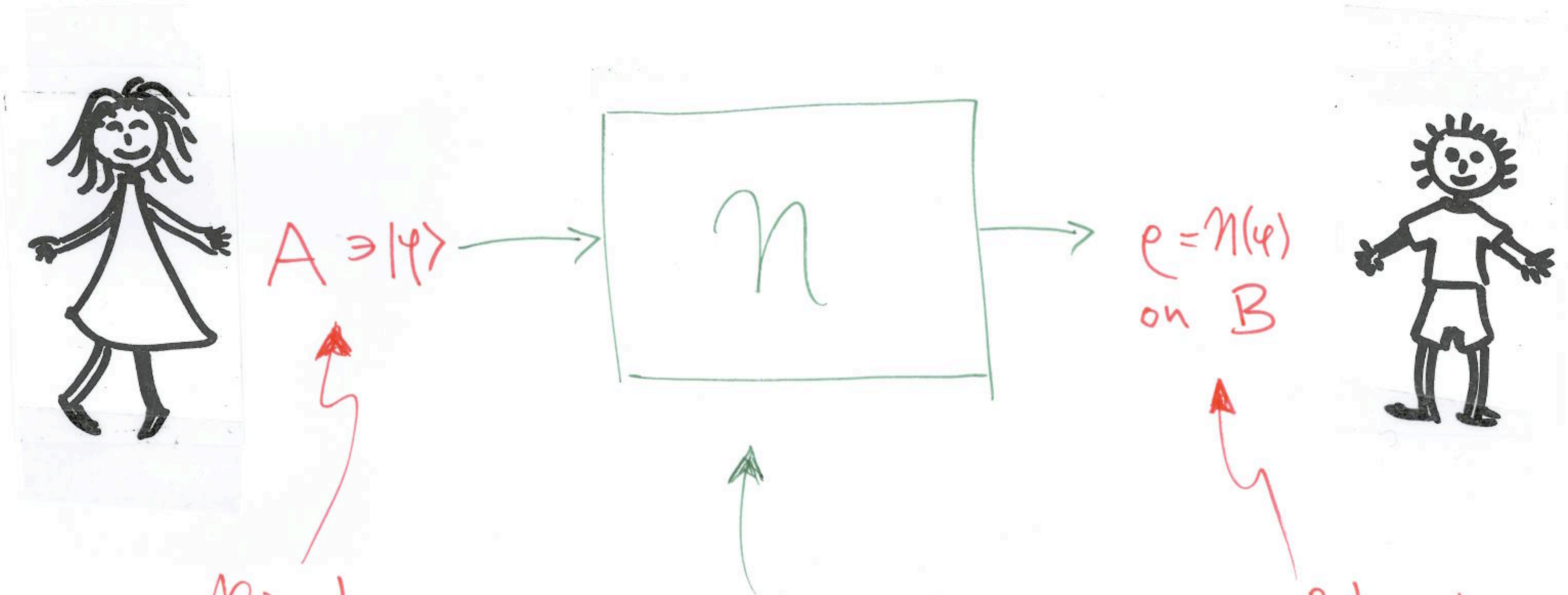
3. Identification (c. + qu.)

4. Quantum-ID ... Fidelity... Forgetfulness

5. Capacity results

6. Outlook

1. What does it mean to transmit a state?



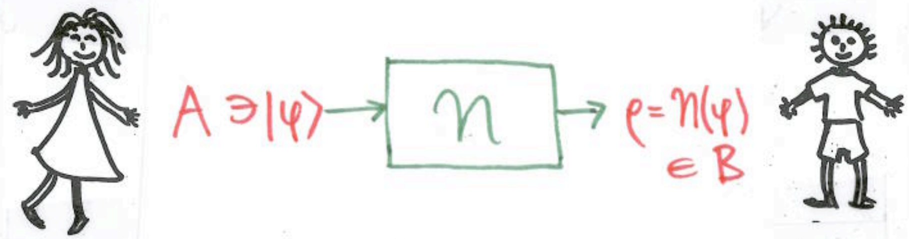
Alice has some quantum state $\varphi = |\varphi\rangle\langle\varphi|$ (only consider pure)

superoperator (completely positive & trace preserving — cptp — map) a.k.a. channel

Bob gets an output state ρ (generally mixed)

*$A, B, \ldots$ have finite dimension $|A|, |B|, \ldots$ throughout

For faithful transmission of Alice's state φ we need:



(1) $B = A$ [to be able to compare ρ to φ]

(2) $\rho = \mathcal{N}(\varphi) = \varphi$ for all $|\varphi\rangle \in A$

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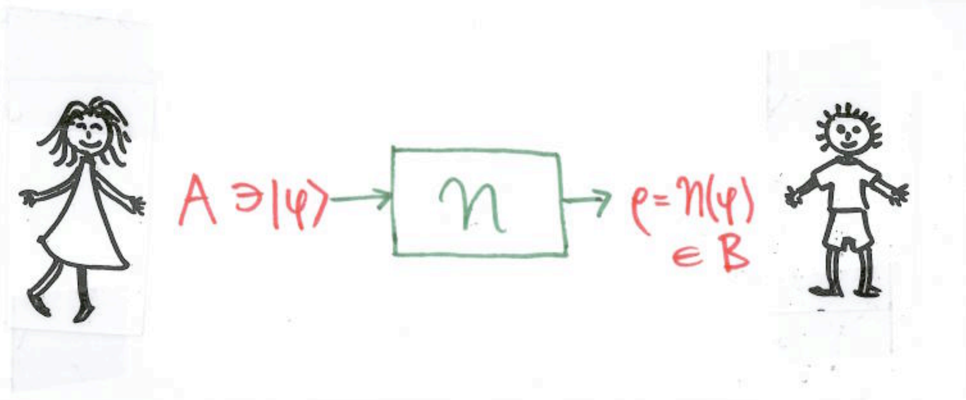
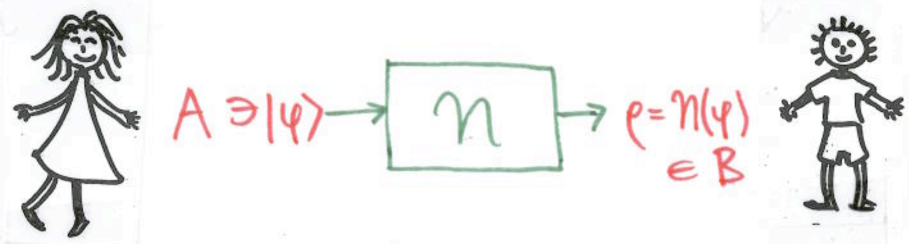


(1) $B = A$ [to be able to compare ρ to φ]

(2) $\rho = \mathcal{N}(\varphi) = \varphi$ for all $|\varphi\rangle \in A$

... but that's perhaps asking too much: only leaves $\mathcal{N} = \mathrm{id}$; and what about small loss of coherence in practice?

For *faithful* transmission of Alice's state φ we need:



(1) $B = A$ [to be able to compare ρ to φ]

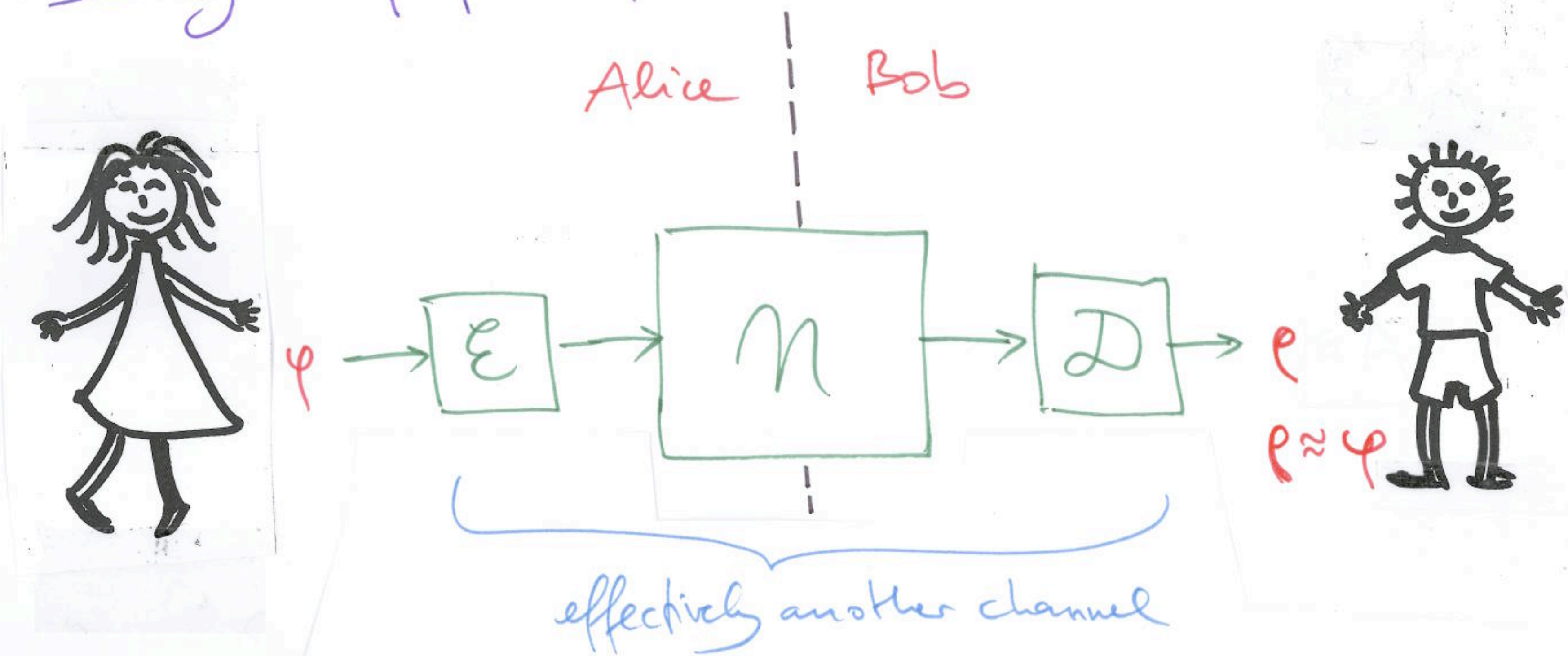
(2) $\rho = \mathcal{N}(\varphi) \approx \varphi$ for all $|\varphi\rangle \in A$

Approximation measured by *fidelity* $F(\varphi, \rho) = \langle\varphi|\rho|\varphi\rangle = \mathrm{Tr}\, \rho\varphi$

[... extends to *mixed state fidelity*
$F(\rho, \sigma) = \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right)^2 = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$]

Condition: $F(\varphi, \rho) \geq 1 - \epsilon$ for all $|\varphi\rangle \in A$

Relaxed condition: there exist encoding and decoding cptp maps $\mathcal{E}$ and $\mathcal{D}$ s.t.



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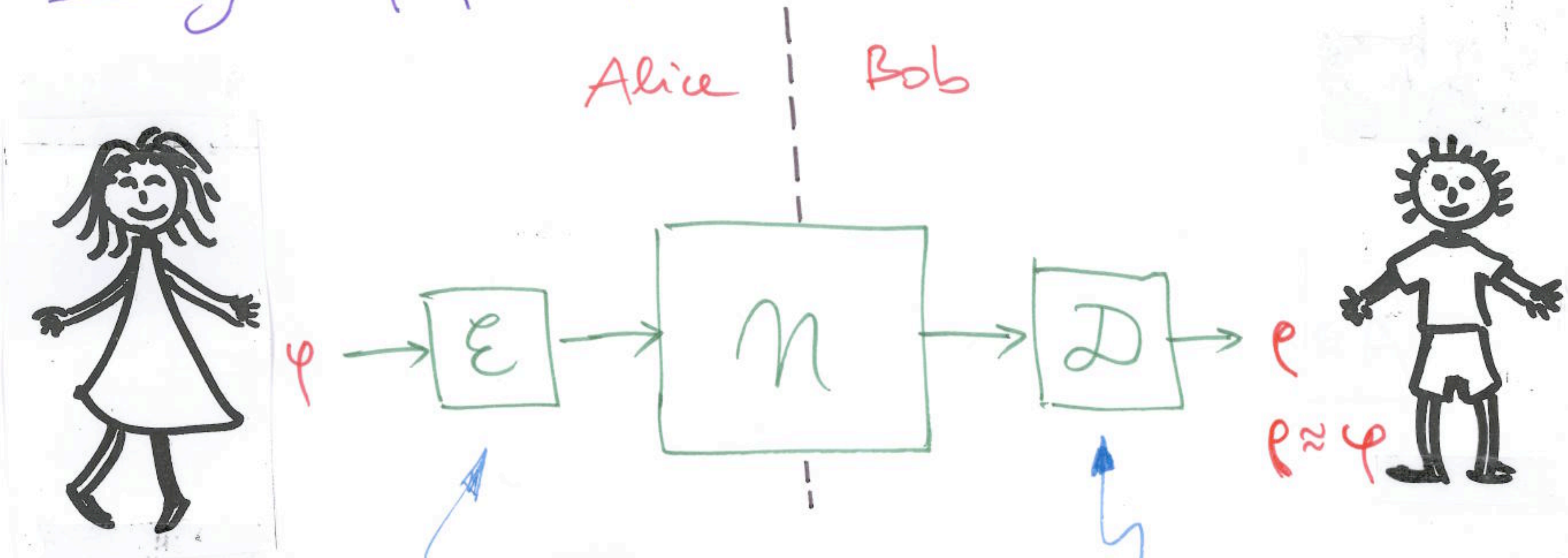
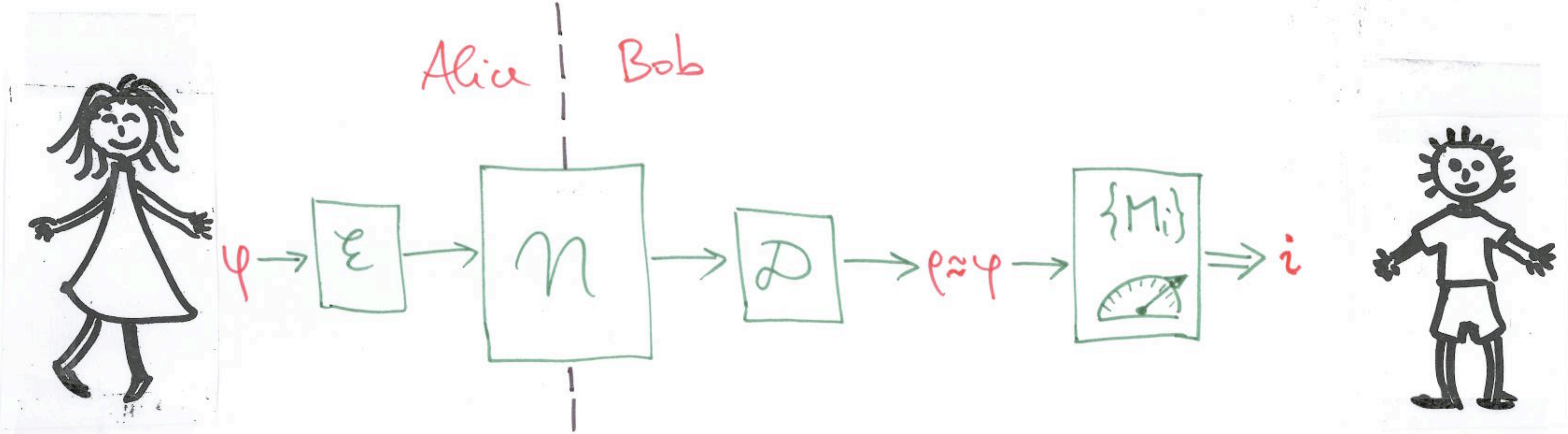


Purpose: to prepare the state for $\mathcal{N}$

This is basically an error correction step.

Also: Allows $B \neq A$

...if this is the case, Bob can perform any measurement (POVM) $\{M_i\}$ [i.e. $M_i \geq 0$, $\sum_i M_i = \mathbb{1}$] on φ:



I.e.,

$$\Pr\{M_i|\varphi\} = \mathrm{Tr}\,\varphi M_i \approx \mathrm{Tr}\,\rho M_i$$

ℓ^1-distance $\leq \delta = 2\sqrt{\epsilon}$

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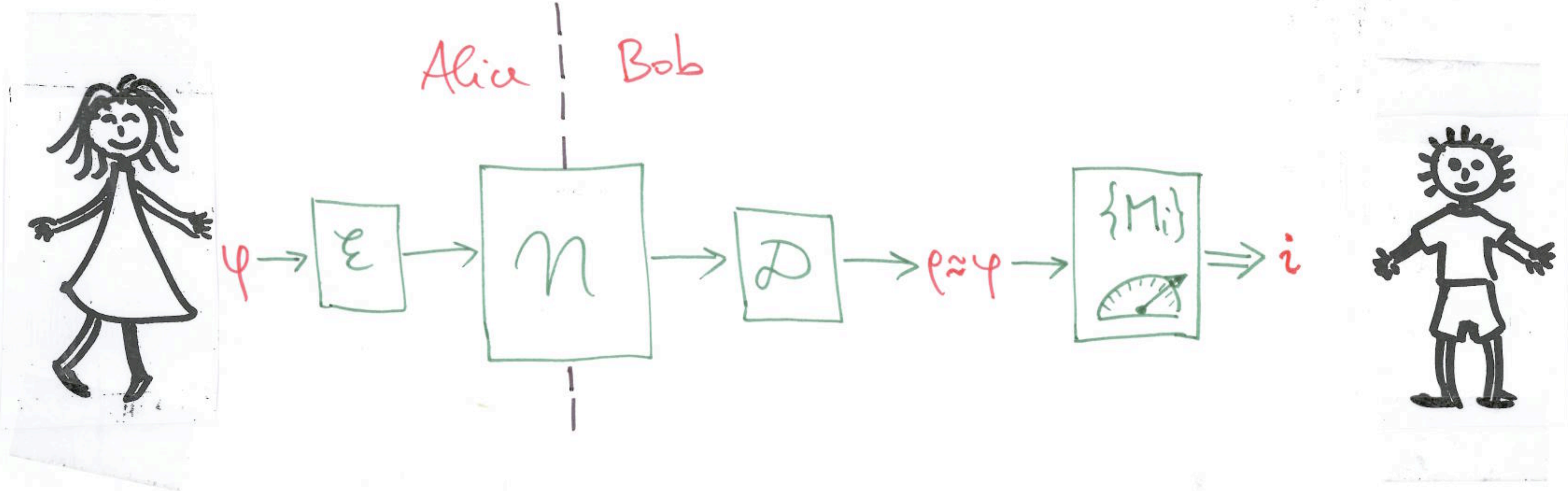

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(Heisenberg picture: action on observables)

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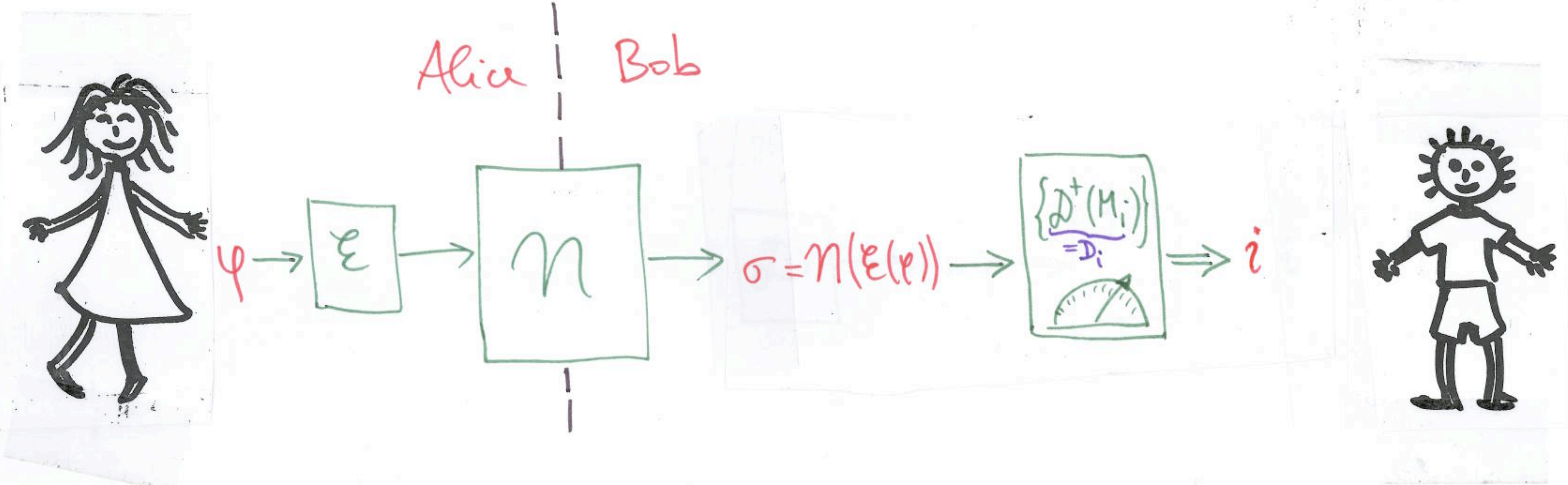

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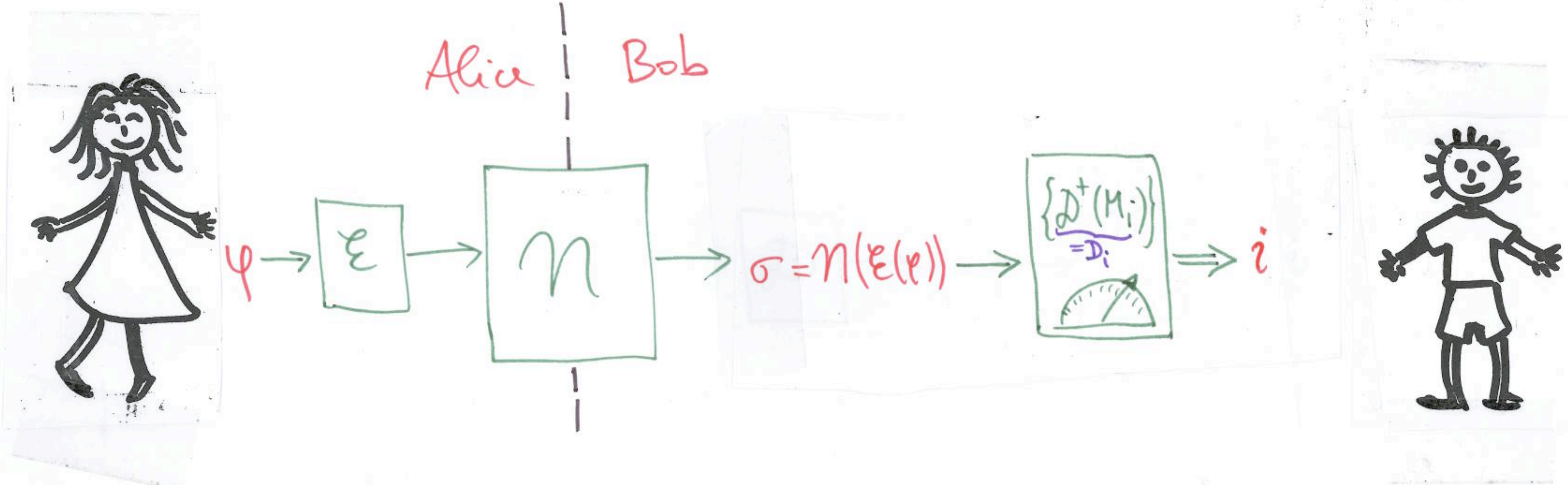

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In other words: By measuring $\{D_i\}$ on σ, Bob can <u>simulate</u> the measurement of $\{M_i\}$ on φ.

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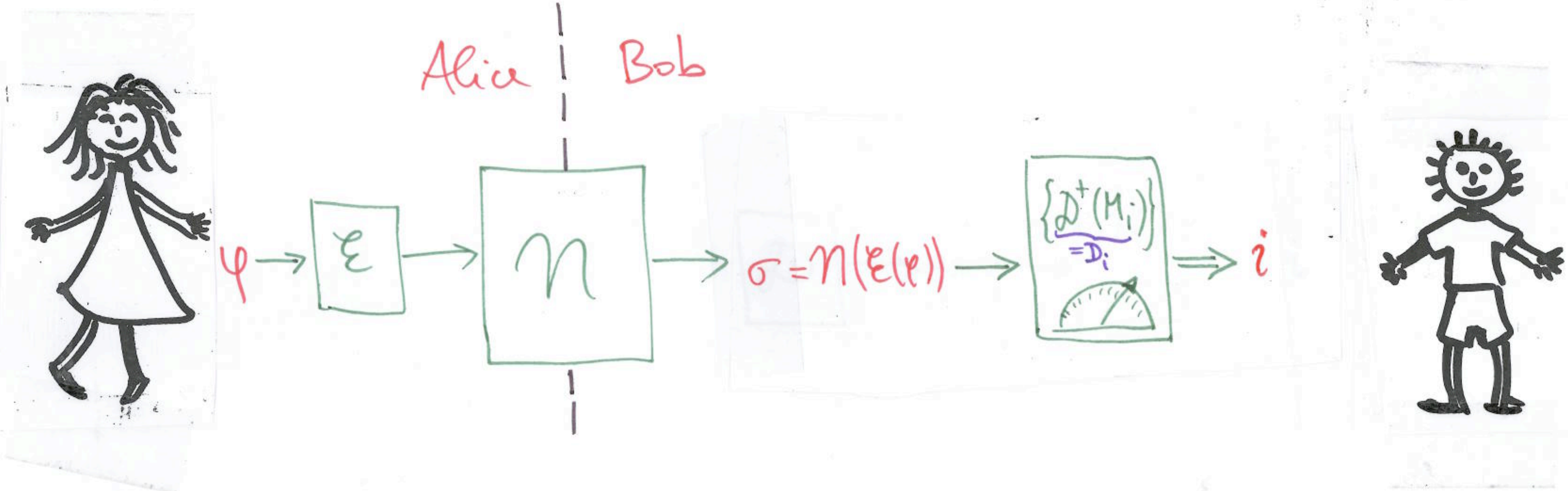
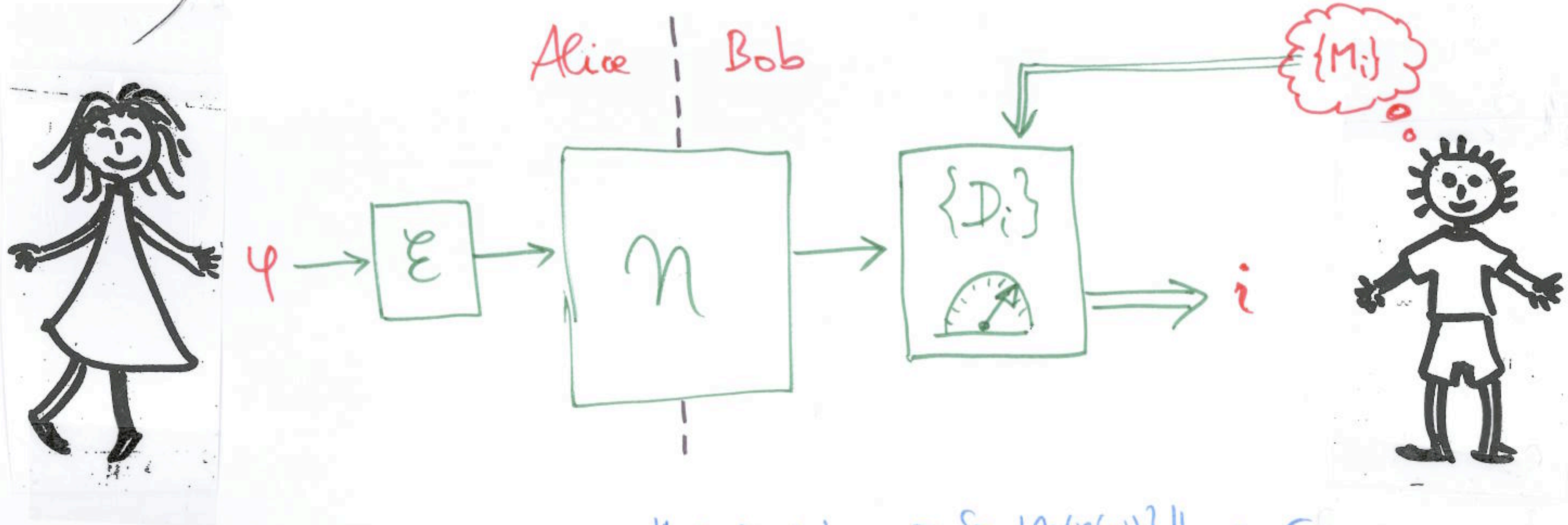


Conversely: If $|\varphi\rangle \in$ d-dim. system, and Bob can accurately simulate the discrete Weyl operators X_d & Z_d, then he can also build a decoder $\mathcal{D}$ s.t.

$\varphi \approx \mathcal{D}(\mathcal{N}(\mathcal{E}(\varphi)))$ [see Hayden/Shor/W., OSID 15(1), 2008
building on Christandl/W., IEEE-IT 51(9), 2005]

2. Simulation of (binary) measurements

Since we can characterize transmission of φ by Bob's ability to simulate measurements on it:



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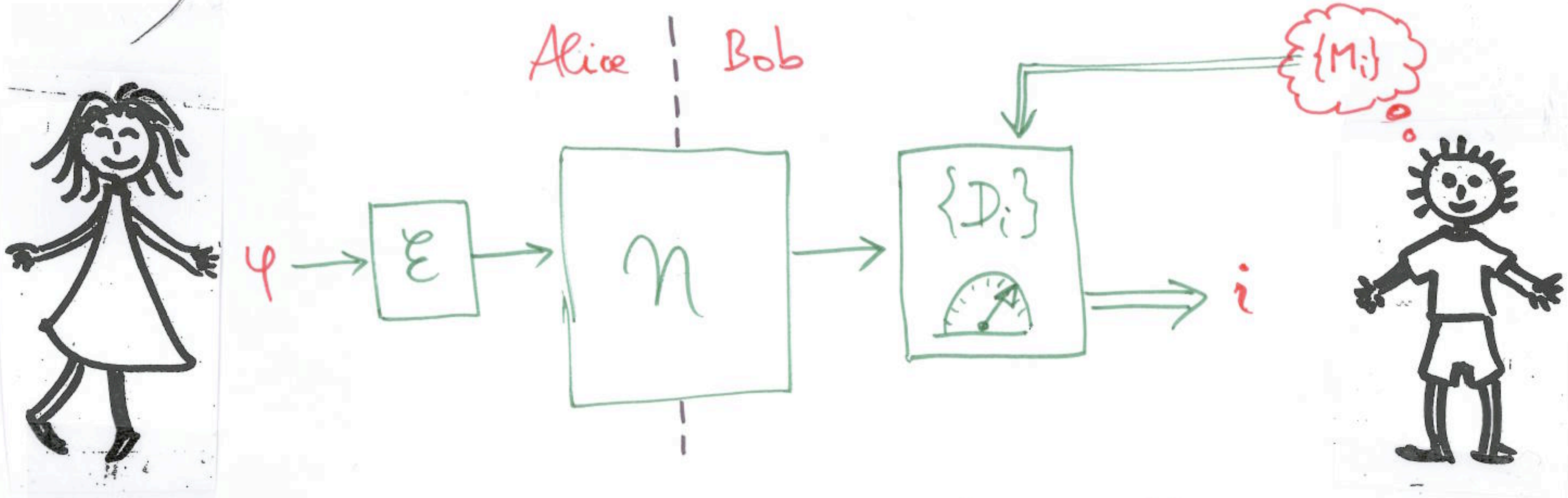
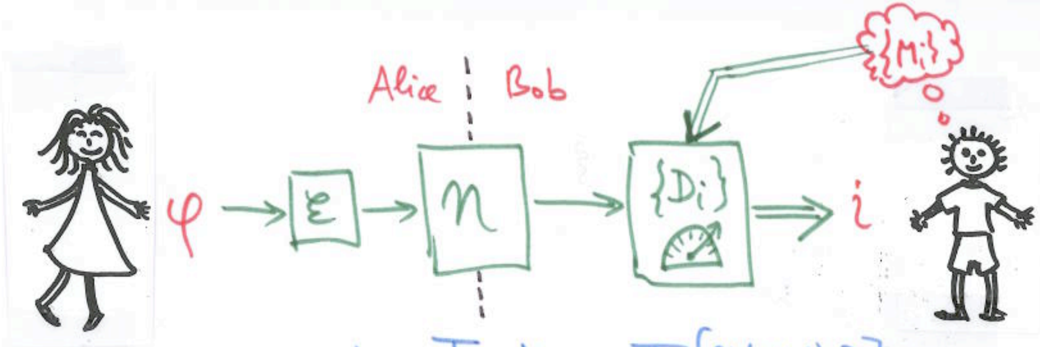


s.t. for all $|\varphi\rangle$, $\quad \left\| \Pr\{M_i | \varphi\} - \Pr\{D_i | \mathcal{N}(\mathcal{E}(\varphi))\} \right\|_1 \leq \epsilon, \ldots$

... we are motivated to modify the game via restrictions on the POVMs we wish to be able to simulate.

Here is what we get for various classes of POVMs $\{M_i\}$:



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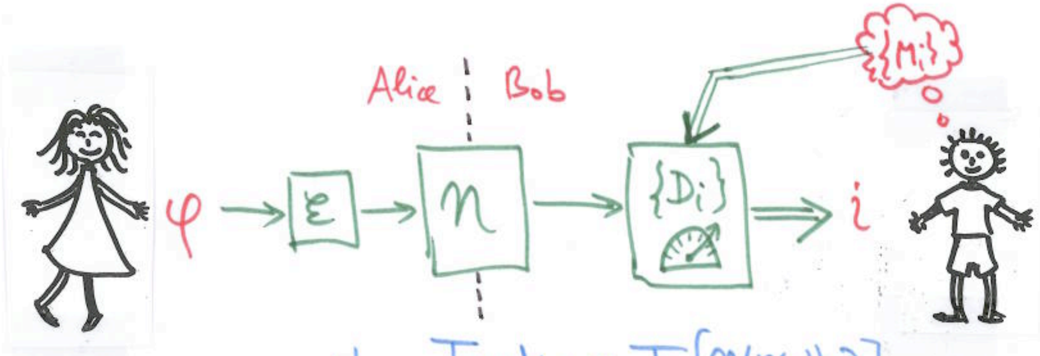

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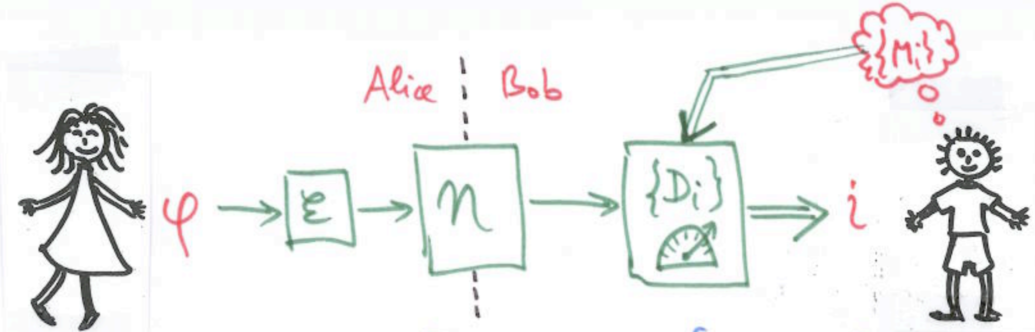

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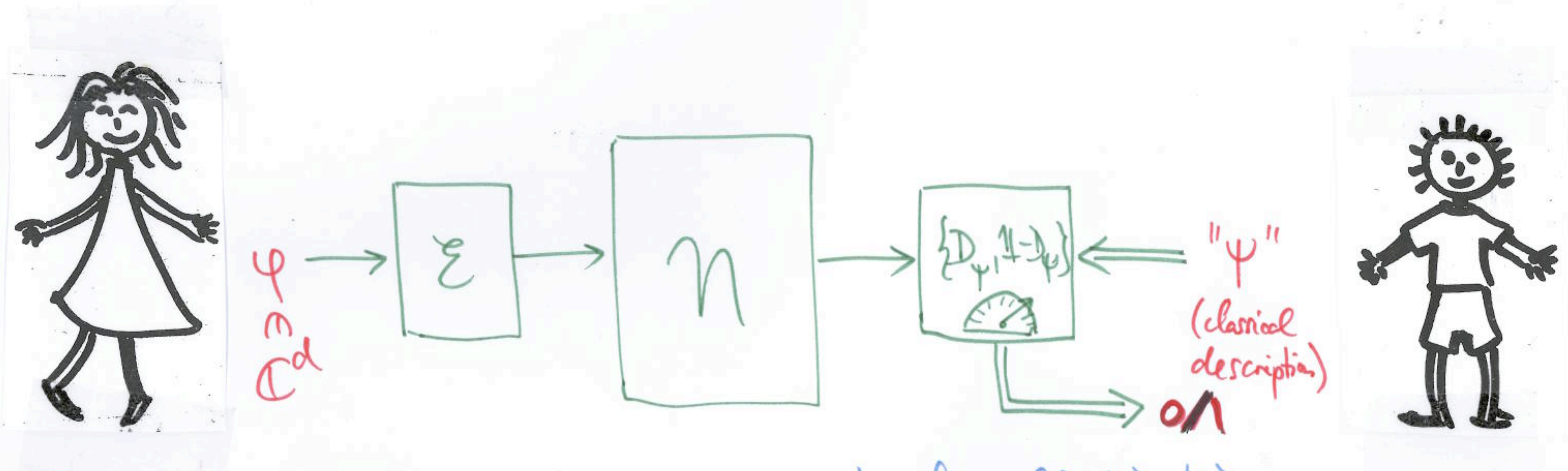
all POVMs of the form $(\lvert\psi\rangle\langle\psi\rvert, \mathbb{1} - \lvert\psi\rangle\langle\psi\rvert)$, $\lvert\psi\rangle$ pure state	quantum identification

3. Identification



s.t. for all $|\varphi\rangle, |\psi\rangle$

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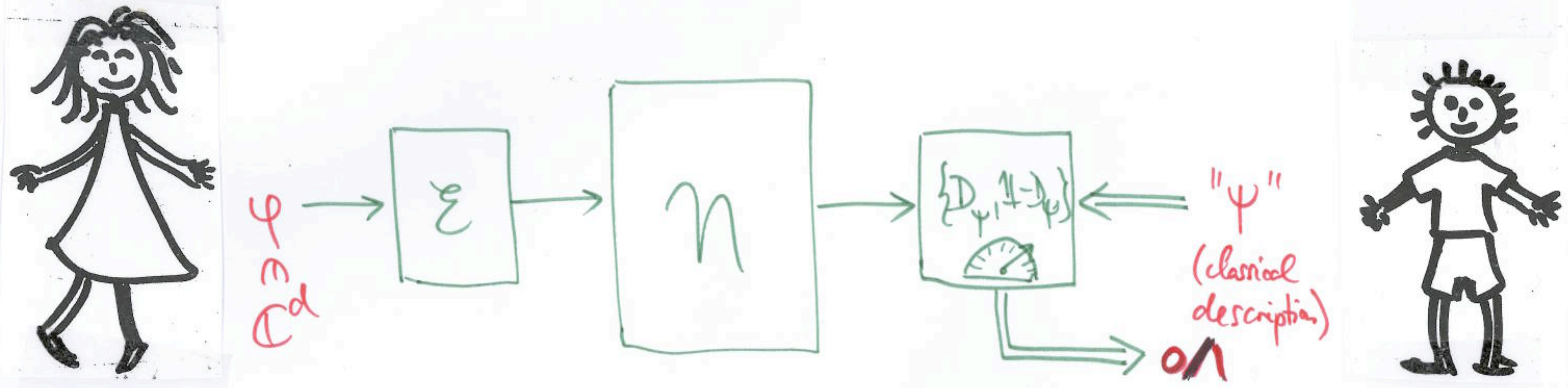

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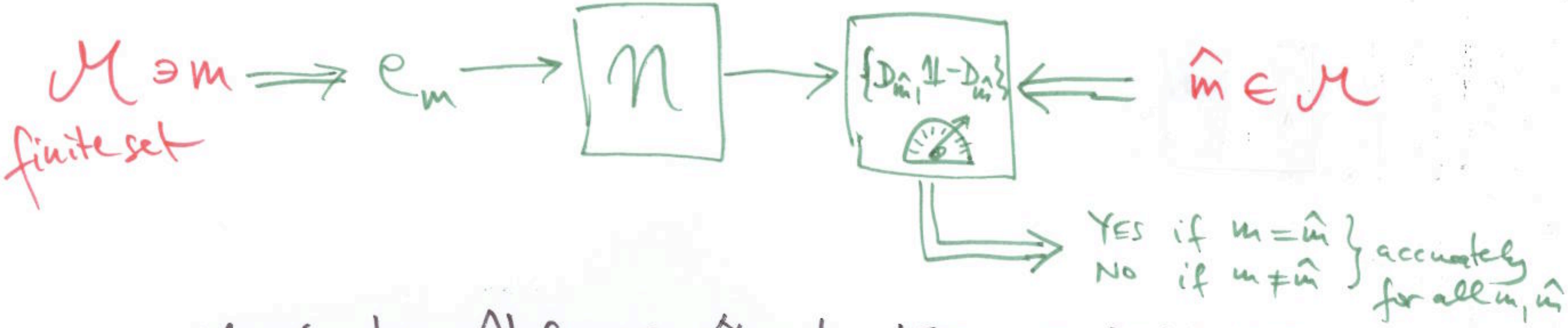
<u>Quantum-ID code</u>: $\mathcal{E}$ and the collection of $0 \leq D_\psi \leq \mathbb{1}$ ($|\psi\rangle \in \mathbb{C}^d$).

<u>Parameters</u>: Error $\underline{\epsilon}$ and encoded dimension $\underline{d}$.

Classical version :



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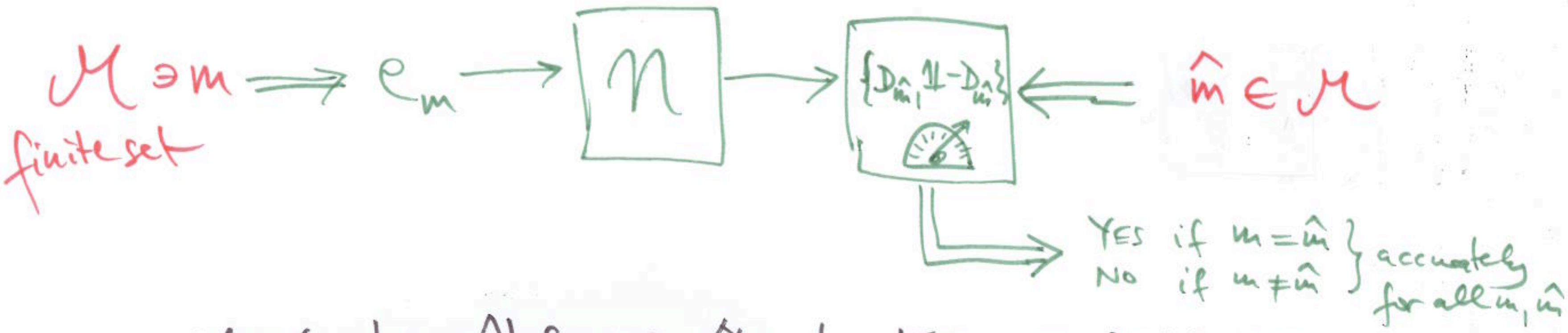

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$\rightsquigarrow$ For n channel uses $\mathcal{N}^{\otimes n}$, $\max|\mathcal{M}| = 2^{2^{C(\mathcal{N})n+o(n)}}$, where $C(\mathcal{N})$ is the Shannon capacity of $\mathcal{N}$.

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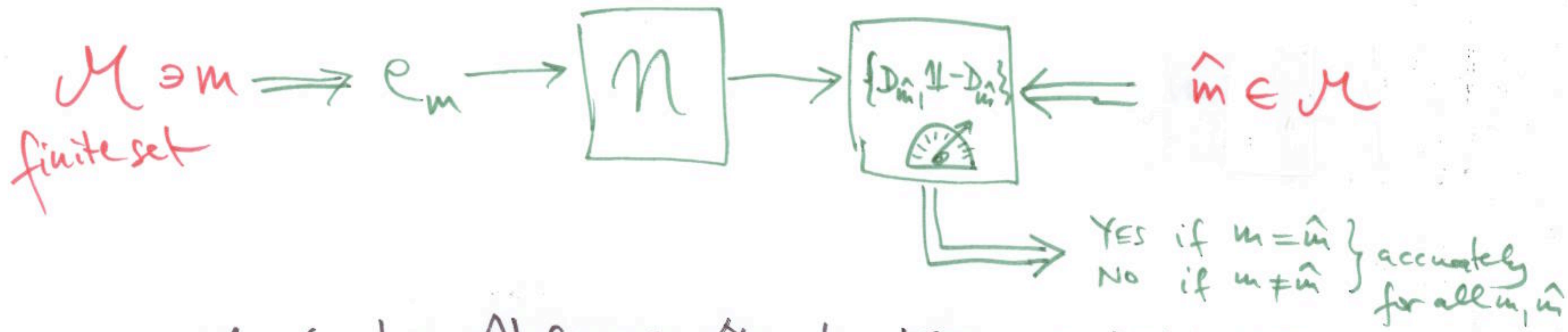

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Löber looked at general quantum channels (PhD thesis, Bielefeld 1999),
but with restrictions on en-/decoding.
Ahlswede/W., IEEE-IT $\underline{48}$(3), 2002 show for cq-channel $\mathcal{N}$, that
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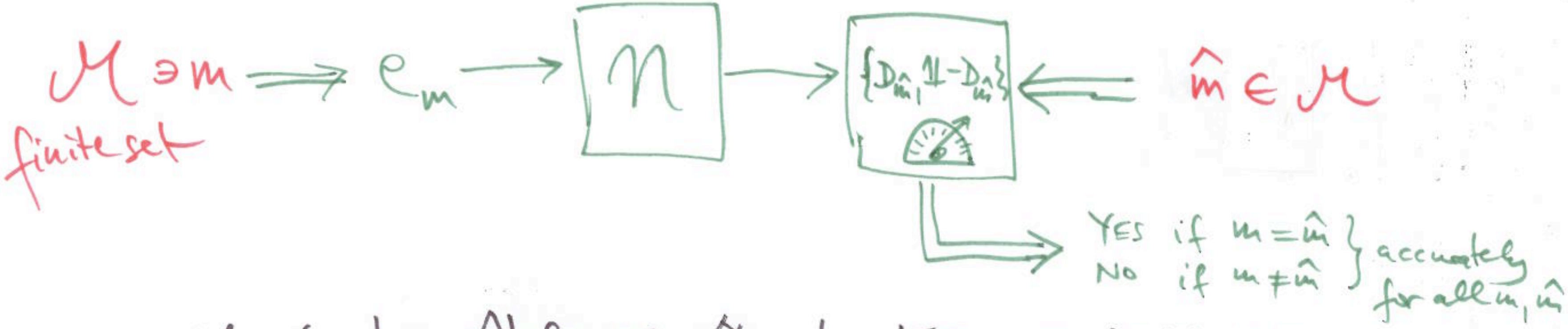

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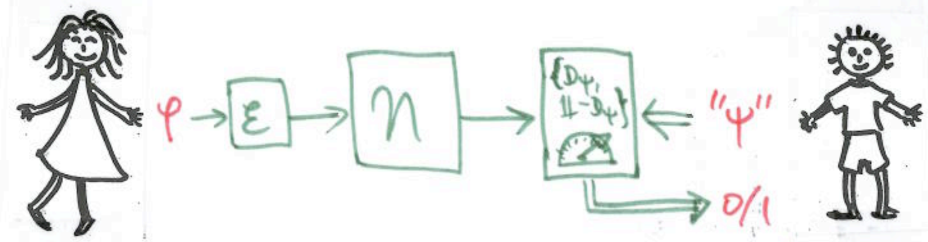
W. ("Holevo 60" festschrift, Rinton 2004) shows: for $\mathcal{N} = id_2$ (noiseless qubit), $\max|\mathcal{M}| = 2^{2^{2n+o(n)}}$

{with pure ρ_m get only $2^{2^{n+o(n)}}$ by "fingerprint" states of Buhrman et al., PRL 2001}

4. Quantum-ID, Fidelity and Forgetfulness



s.t. $\operatorname{Tr} \varphi\psi \approx \operatorname{Tr}[\mathcal{N}(\mathcal{E}(\varphi)) D_\psi]$

1st observation: $\mathcal{N} \circ \mathcal{E}$ is δ-geometry preserving $[\delta = \delta(\epsilon)]$ for pretty much any distance measure of states that is monotonic under cptp maps; incl. fidelity $F(\rho,\sigma)$ and trace distance $\|\rho-\sigma\|_1$.

E.g. $\left| F(\varphi,\psi) - F(\mathcal{N}(\mathcal{E}(\varphi)), \mathcal{N}(\mathcal{E}(\psi))) \right| \leq \delta$

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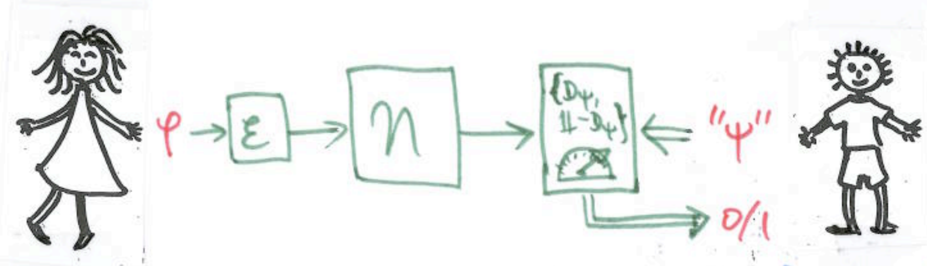

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Intuitive reason: on $\mathrm{span}\{|\varphi\rangle, |\psi\rangle\} =: Q < A$ we can simulate any measurement $\Longrightarrow$ hence it is an error correcting code and we can construct a decoding $\mathcal{D}$:

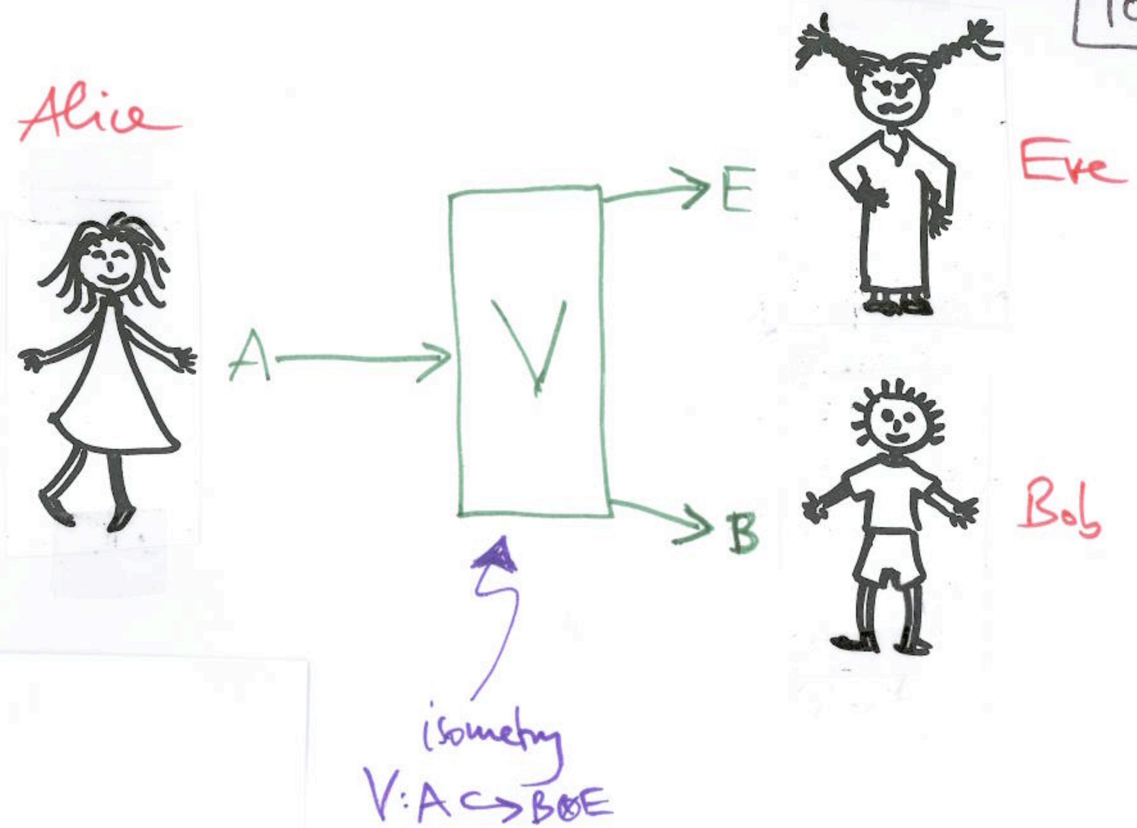
$$\varphi \longmapsto \mathcal{N}(\mathcal{E}(\varphi)) \longmapsto \approx \varphi$$
$$\psi \longmapsto \mathcal{N}(\mathcal{E}(\psi)) \longmapsto \approx \psi$$

... now monotonicity ...

For the next steps contract $\mathcal{N} \circ \mathcal{E}$ to one channel — still called $\mathcal{N}$.

We derive further insights from the Stinespring dilation of it:

$$\mathcal{N}(\varrho) = \mathrm{Tr}_E V \varrho V^\dagger$$



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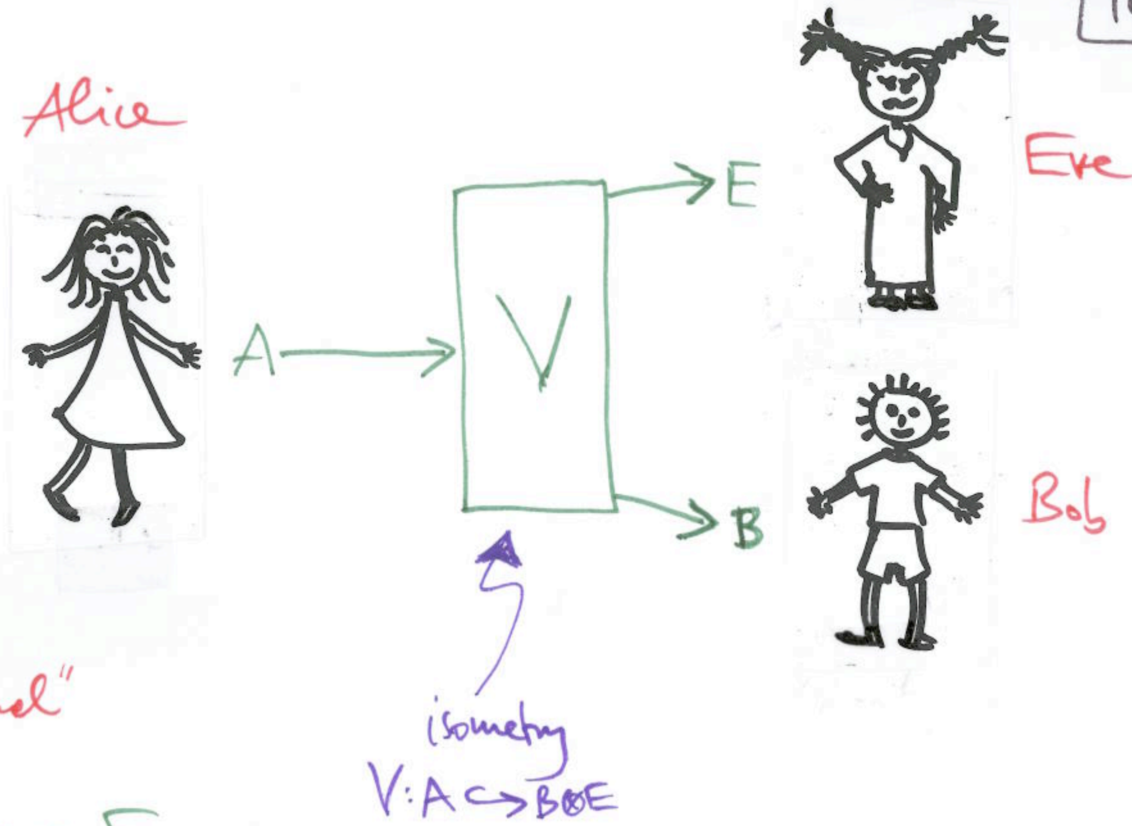
$$\mathcal{N}^c(\rho) = \mathrm{Tr}_B\, V\rho V^\dagger \quad \text{"complementary channel"}$$



- $\mathcal{N}^c$ essentially unique up to unitaries on E
- V allows us to identify the channel $\mathcal{N}$ with a subspace $S := VA \subset B\otimes E$ of the combined Bob+Eve system

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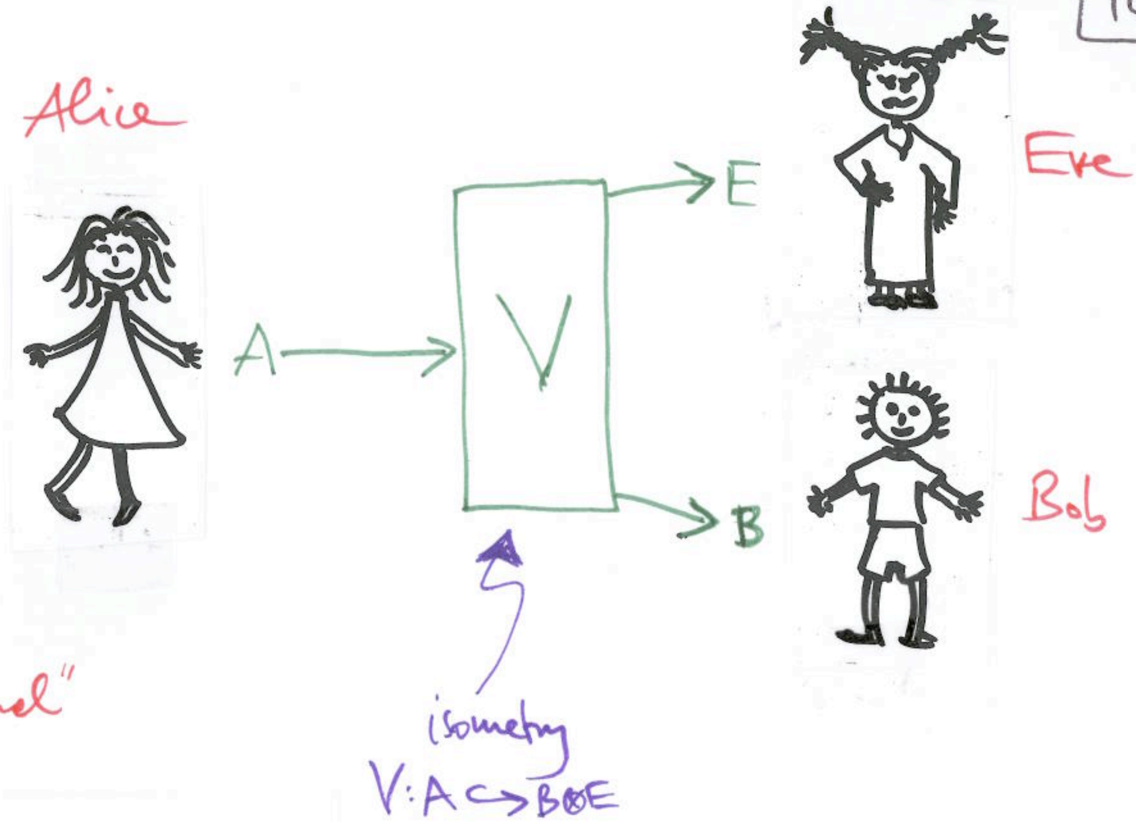

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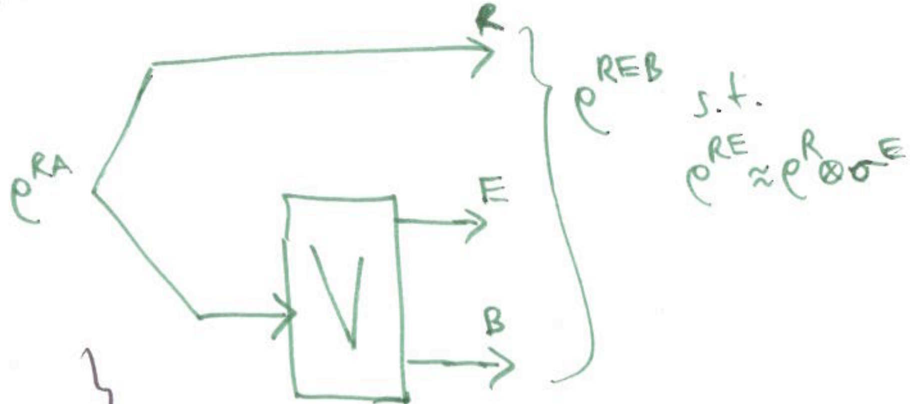
$$\mathcal{N}^c(\rho) = \mathrm{Tr}_B\, V\rho V^\dagger$$ "complementary channel"

Recall the important decoupling principle:

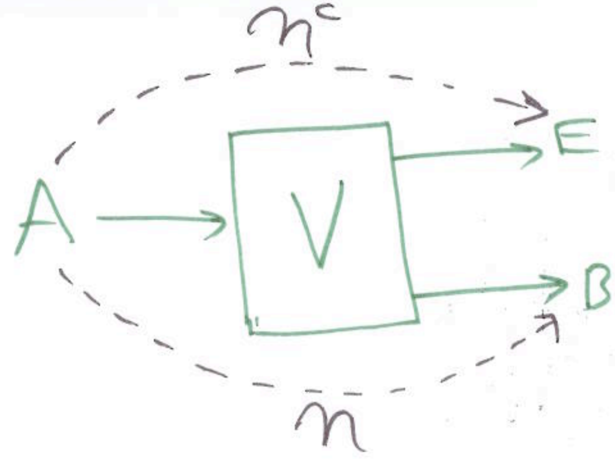
$\mathcal{N}$ is (approx.) correctable [i.e. $\exists\, \mathcal{D}$ cptp s.t. $\mathcal{D}\circ\mathcal{N} \approx \mathrm{id}_A$]

$\Updownarrow$

$\mathcal{N}^c$ is completely forgetful [i.e. $\forall \rho^{RA}$ $(\mathrm{id}\otimes\mathcal{N}^c)\rho^{RA} \approx \rho^R \otimes \sigma^E$]



{Schumacher/Westmoreland, QIP 1(1+2), 2002.
Kretschmann/Schlingemann/Werner, IEEE-IT 54(4), 2008.}



2nd (Fidelity-Alternative): $\mathcal{N}$ is ϵ-fidelity preserving iff $\mathcal{N}^c$ is δ-forgetful [i.e. for all ρ^A on A, $F(\mathcal{N}^c(\rho^A), \sigma^E) \geq 1-\delta$]

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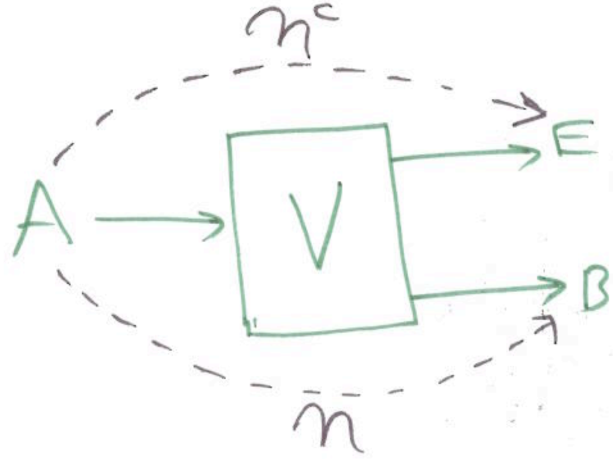
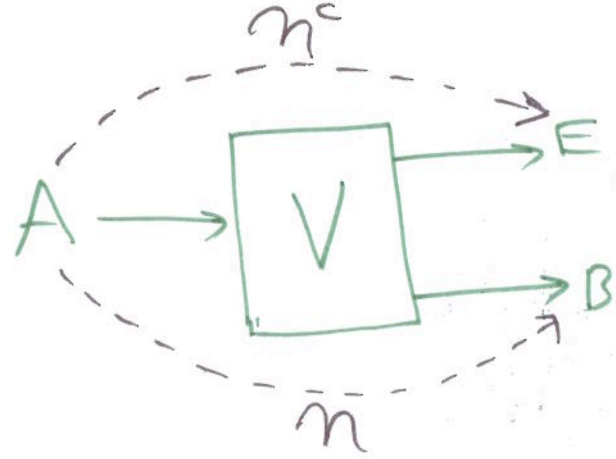


{Intuitively: clear by applying decoupling principle to the 2-dim. subspaces spanned by $|\varphi\rangle, |\psi\rangle$... }



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3rd: If $\mathcal{N}$ is ϵ-fidelity preserving and all states $\mathcal{N}(\varphi)$ [for pure $\varphi = |\varphi\rangle\langle\varphi|$] have sufficiently "flat" spectrum [non-zero eigenvalue range from μ to λ], then one can complete the channel to a q-ID code with error $\eta \leq O\left(\frac{\lambda}{\mu}\right) \epsilon^{O(1)}$ [by constructing suitable operators $0 \leq D_\psi \leq \mathbb{1}$ for $|\psi\rangle \in A$].

In summary:

Quantum-ID code for channel $\mathcal{N}$ $\xLeftrightarrow[\text{(more or less)}]{}$ $\mathcal{N}$ (approx.) geometry preserving $\Longleftrightarrow$ $\mathcal{N}^c$ (approx.) forgetful

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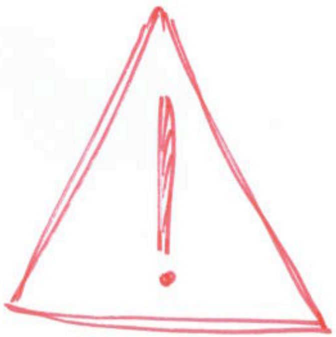
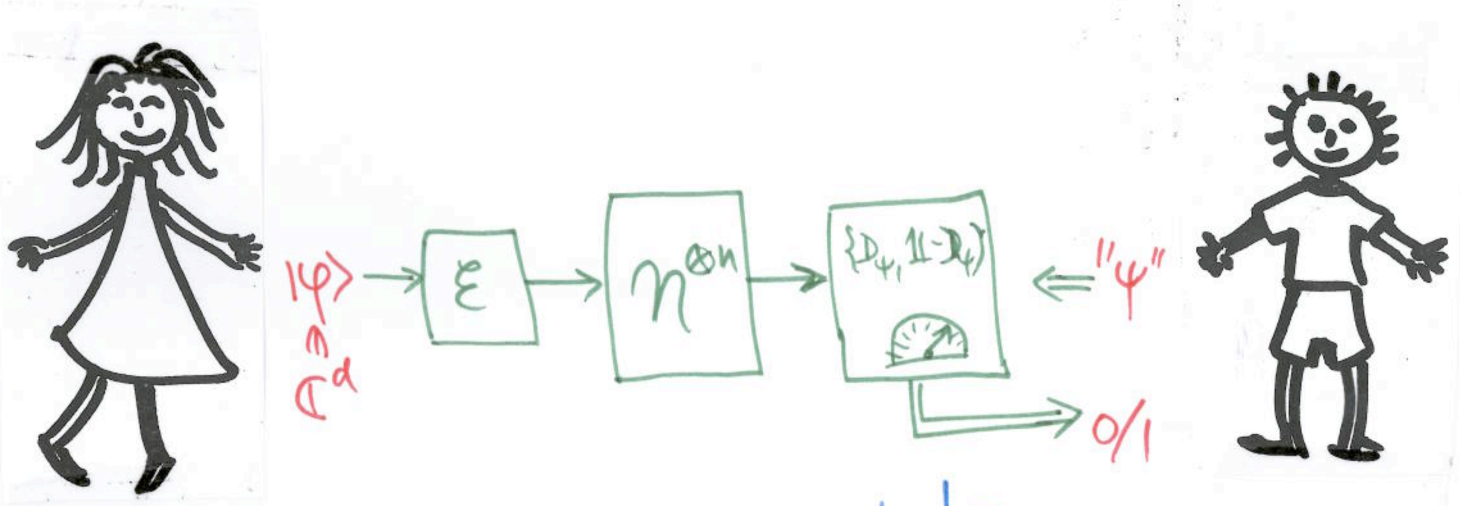


Big difference between forgetfulness and complete forgetfulness (cf. difference between naive norm & diamond norm on superoperators): e.g. consider the $d \to d$ channel $\rho \longmapsto \frac{d\mathbb{1} - \rho^T}{d^2-1}$ or examples in Bennett et al., IEEE-IT 51(1), 2005.

5. Quantum-ID capacity

Now consider using the same channel n times, as $\epsilon \to 0$ and with maximum dimension d:



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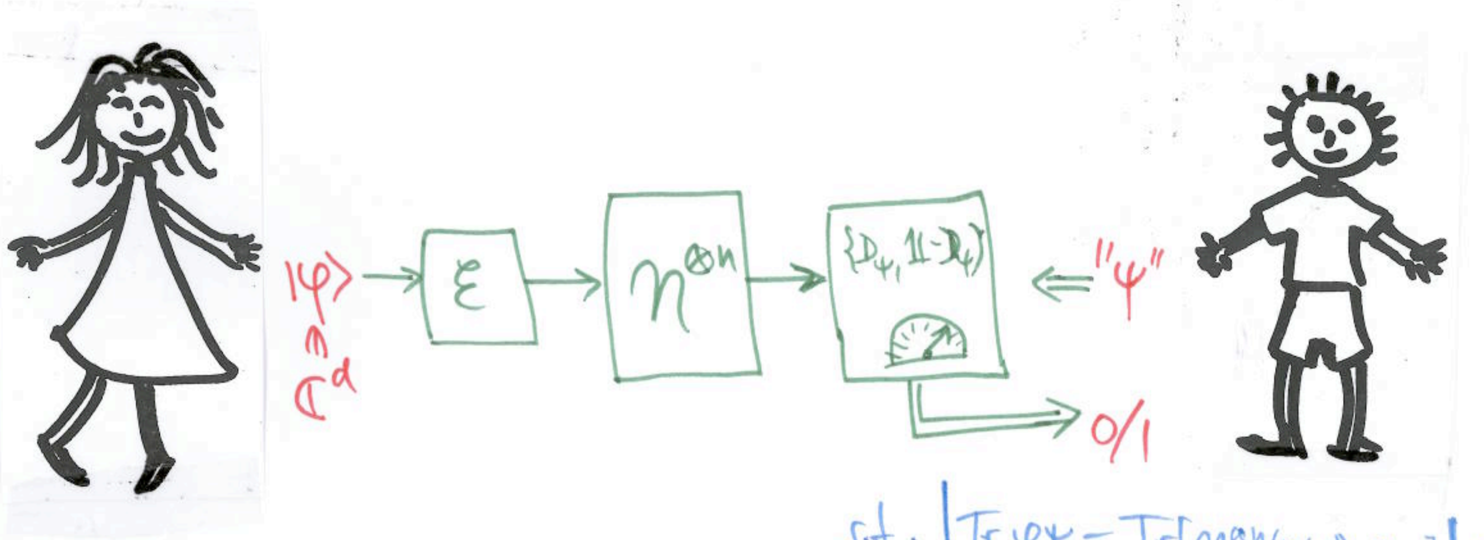

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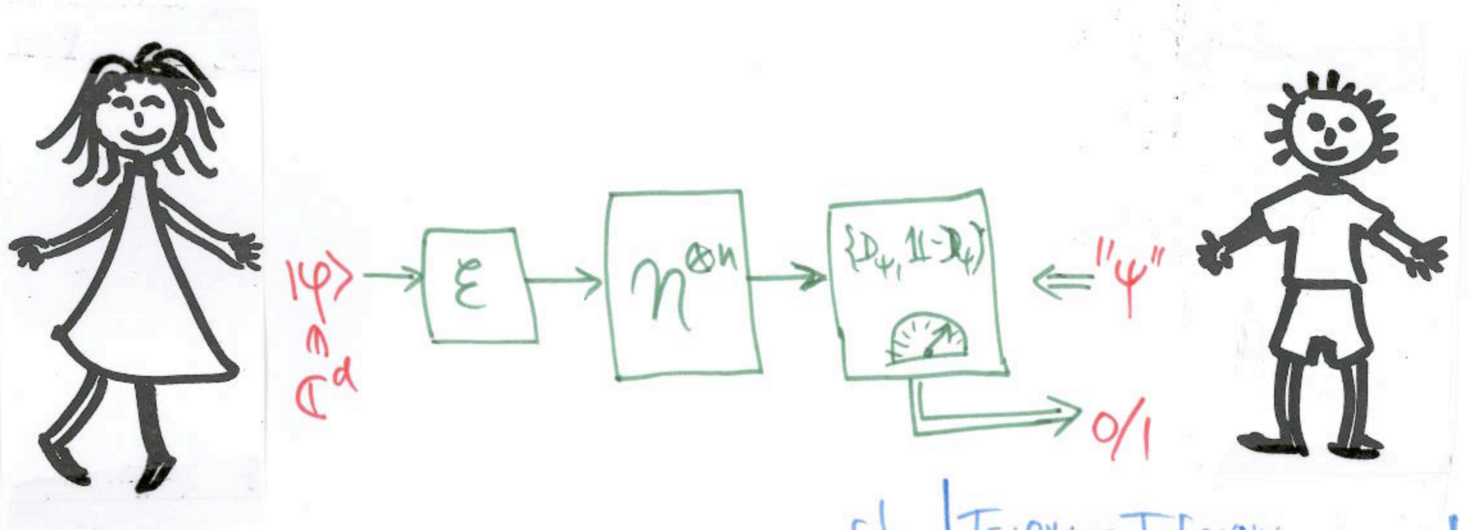

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Hence the <u>definition</u>: the rate of the above code is $\frac{1}{n}\log d =: R$. The maximum achievable rate R s.t. $\epsilon \to 0$ as $n \to \infty$ is the <u>quantum-ID capacity $Q_{ID}(\eta)$</u>.

What do we know about $Q_{ID}(\mathcal{N})$?

(i) From the Fidelity Alternative: $Q_{ID}(\mathcal{N})>0 \Rightarrow Q(\mathcal{N})>0$

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(iii) NEW results — using the Fidelity Alternative we can prove capacity results by forcing forgetfulness for Eve....

Theorem A: The quantum-ID capacity of $\mathcal{N}$ is the regularization of $Q^{(1)}_{ID}(\mathcal{N}) = \sup\left\{ I(A:B) \mid \rho^{AB} = (id \otimes \mathcal{N})\sigma^{AA'} \text{ and } I(A\rangle B) = S(B) - S(AB) > 0 \right\}$

I.e., $Q_{ID}(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} Q^{(1)}_{ID}(\mathcal{N}^{\otimes n})$.

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Amortised rate of a code on $\mathcal{N}^{\otimes n}$, using rn qubits extra: $\frac{1}{n}\log d - r =: \tilde{R}$;

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⚠ equals the entanglement-assisted classical capacity of $\mathcal{N}$! [Bennett et al., IEEE-IT 48(10), 2002]

Ideas for the proof — both Thm. A & B:

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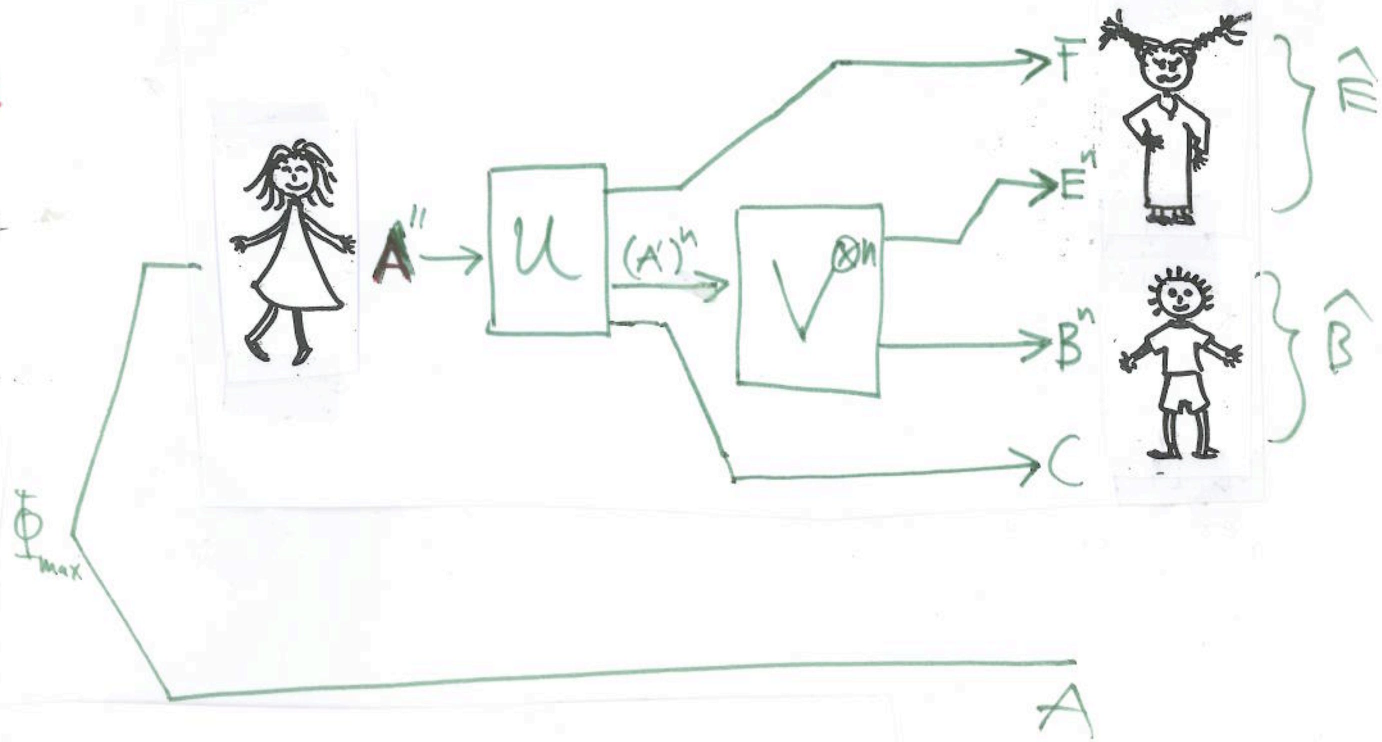
The typical q-ID code on the right maps the encoded space A to a subspace $S \subset \hat{B} \otimes \hat{E}$.



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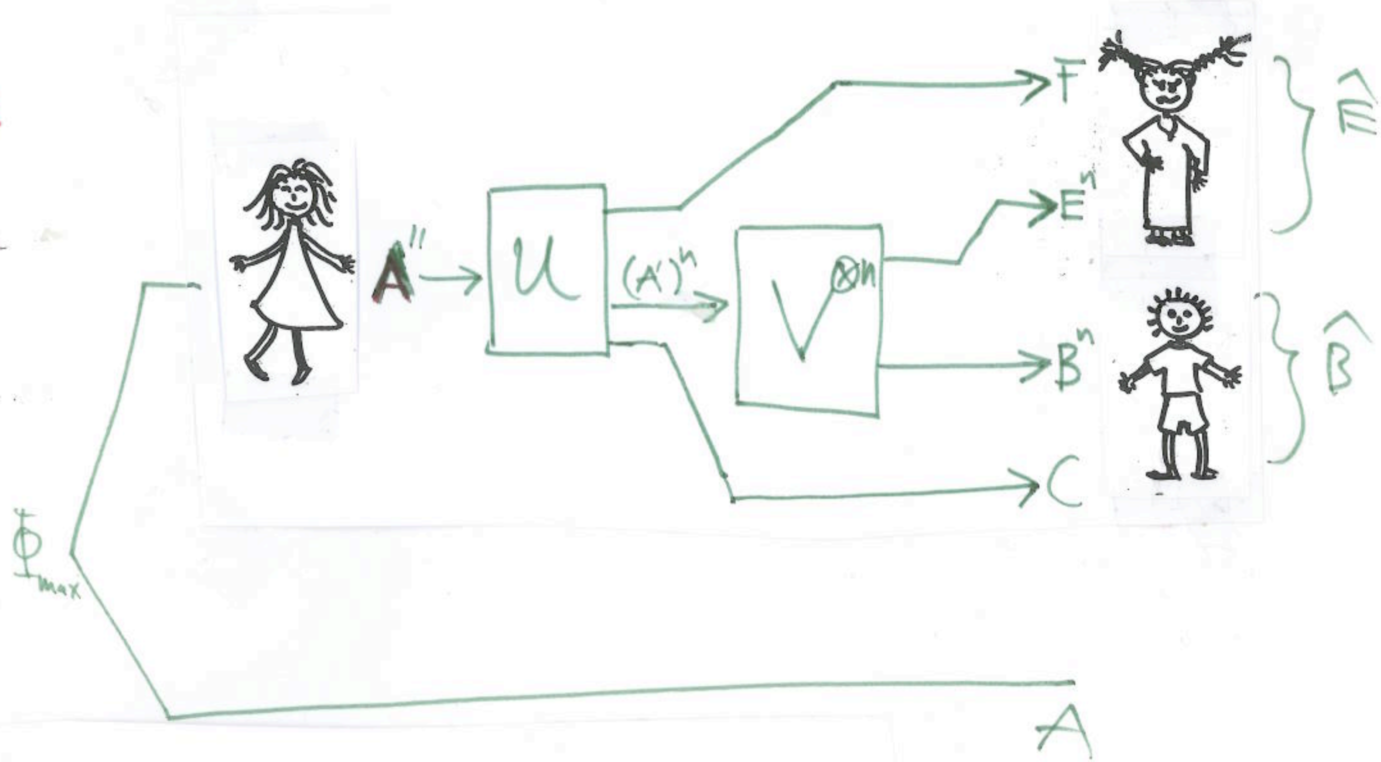

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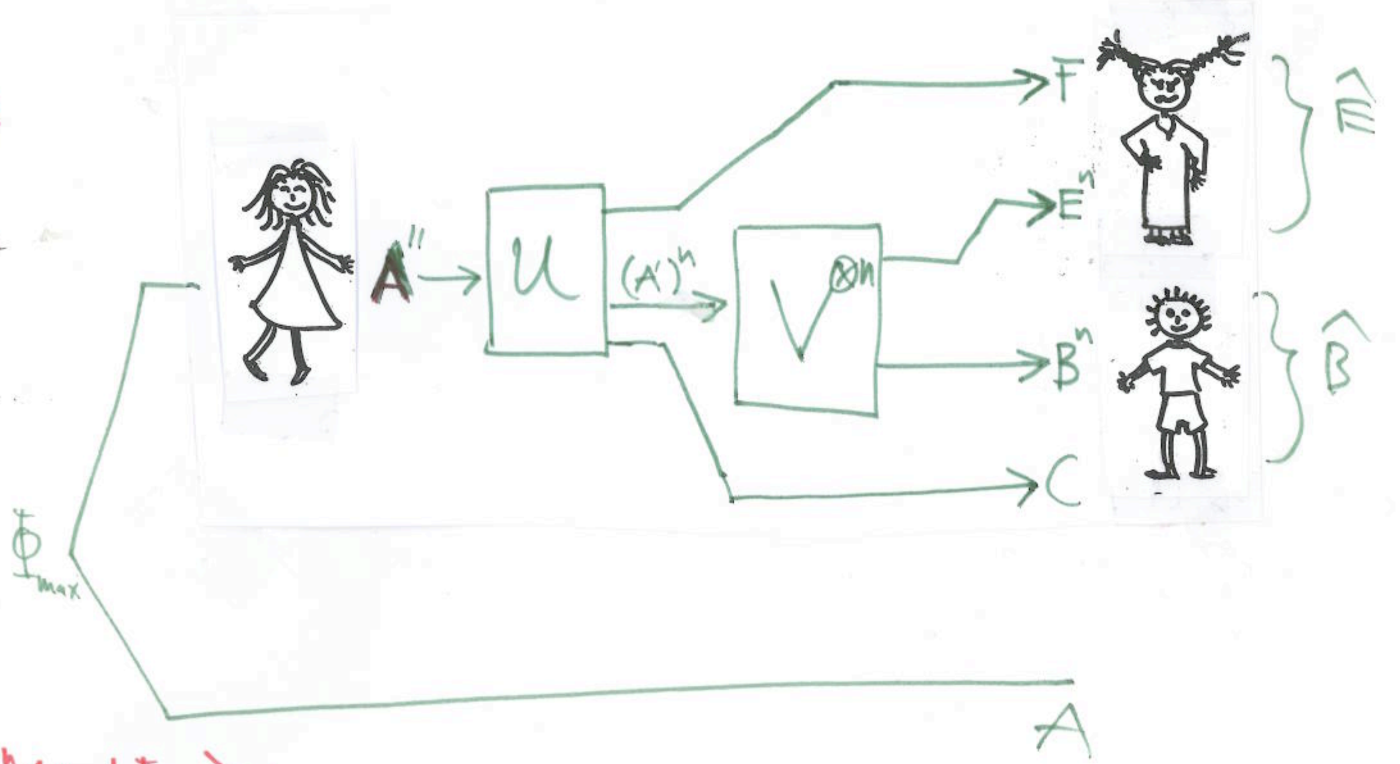

Now, w.r.t. the state $|\psi\rangle = V^{\otimes n}\,\mathcal{U}\,|\Phi_{max}\rangle$:

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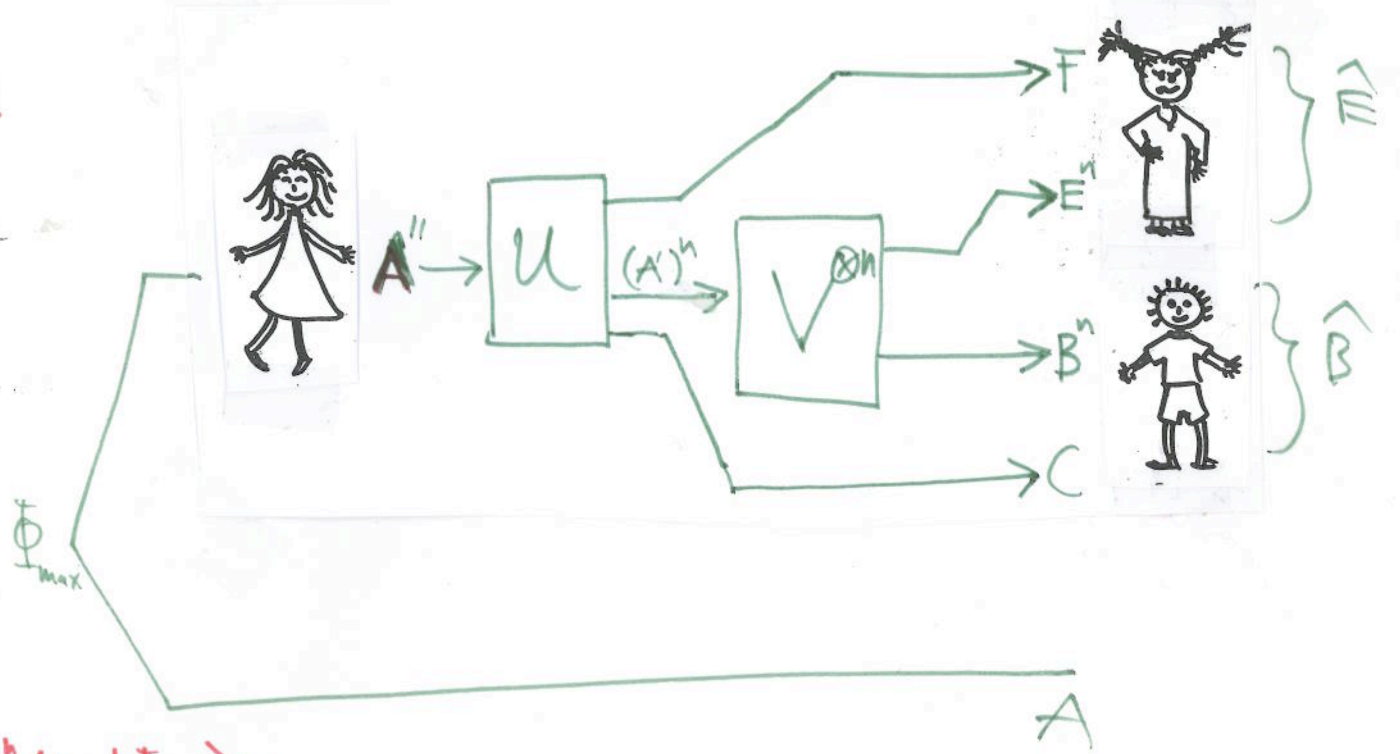

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At the same time, for $|C| = 1$:

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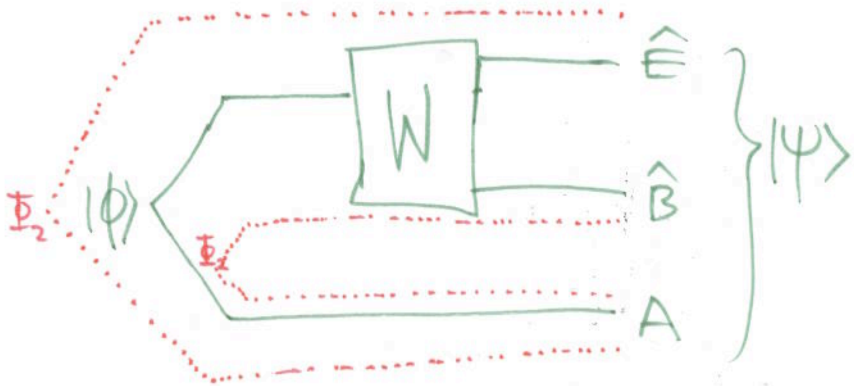

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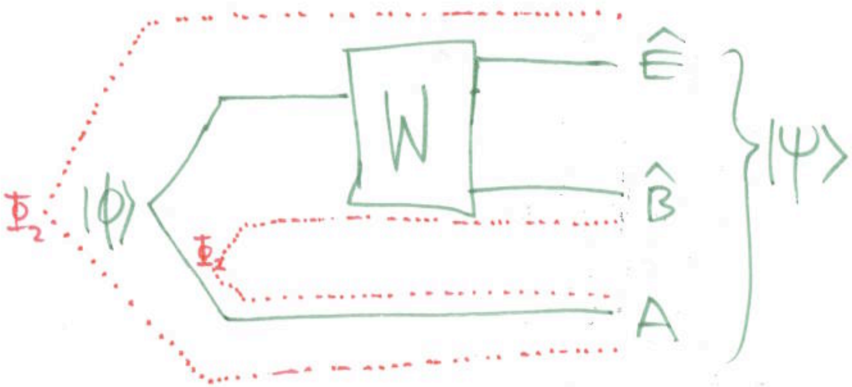

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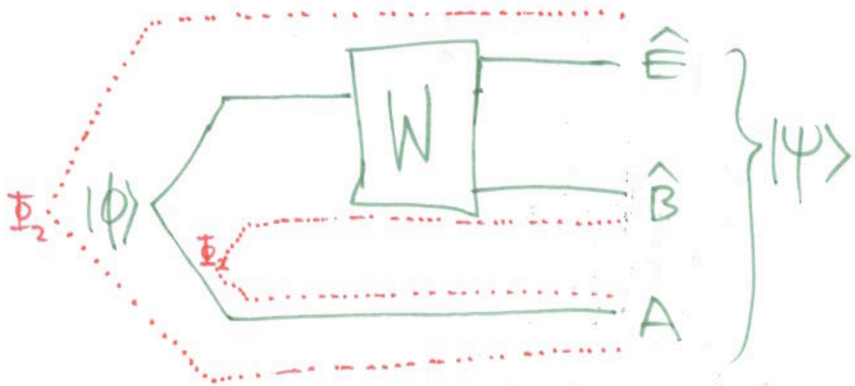

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