Most quantum states are useless for measurement-based quantum computation

Steve Flammia

Perimeter Institute

QIP 2009, Santa Fe

D. Gross, SF, J. Eisert 0810.4331

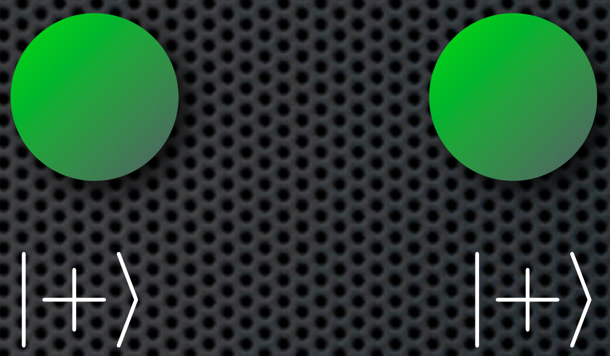
M. Bremner, C. Mora, A. Winter 0812.3001

Measurement-based QC

Raussendorf & Briegel PRL 2001

Measurement-based QC

- prepare X eigenstates



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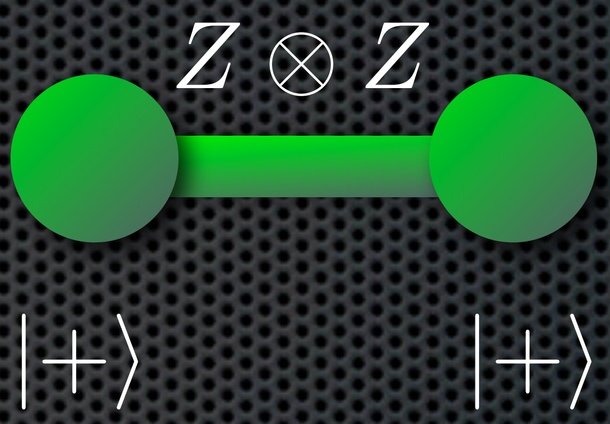

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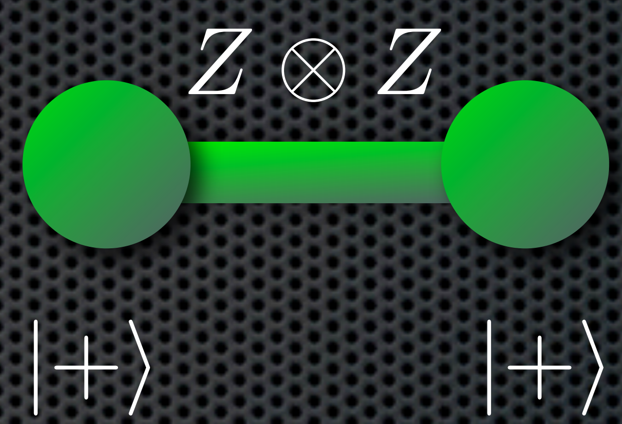

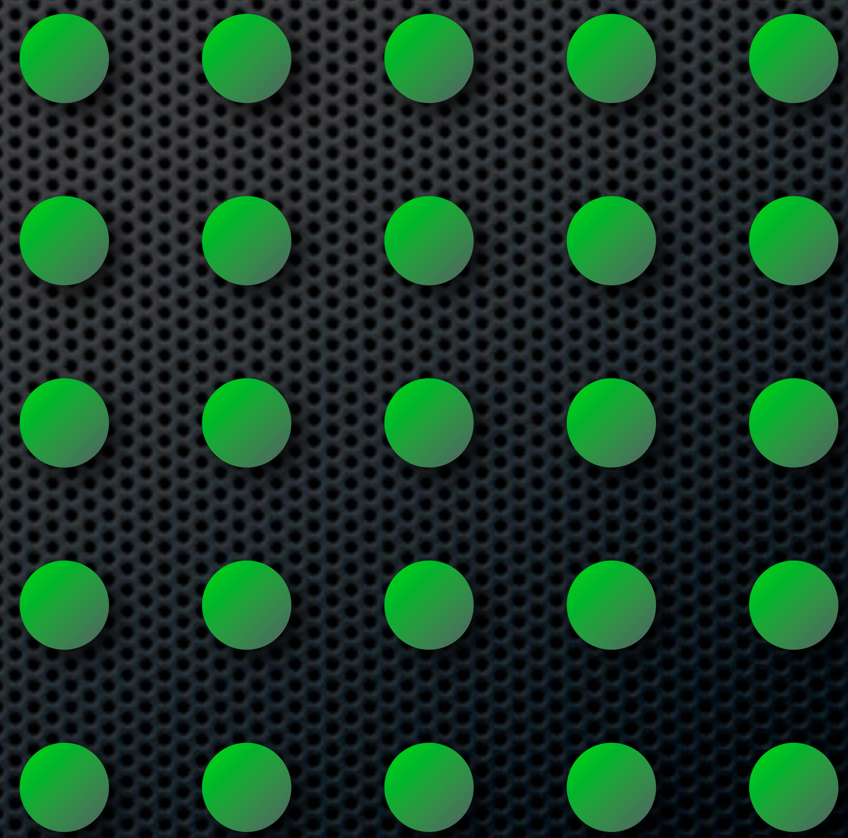

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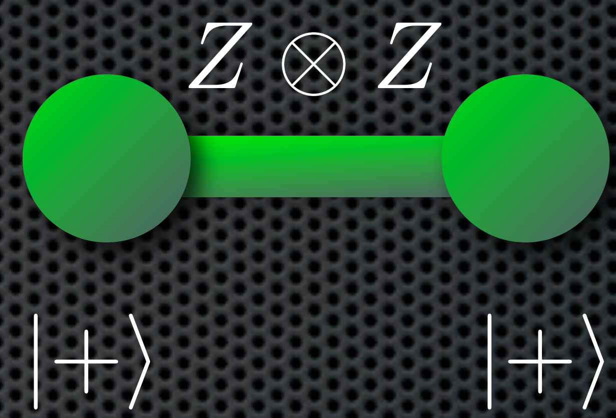

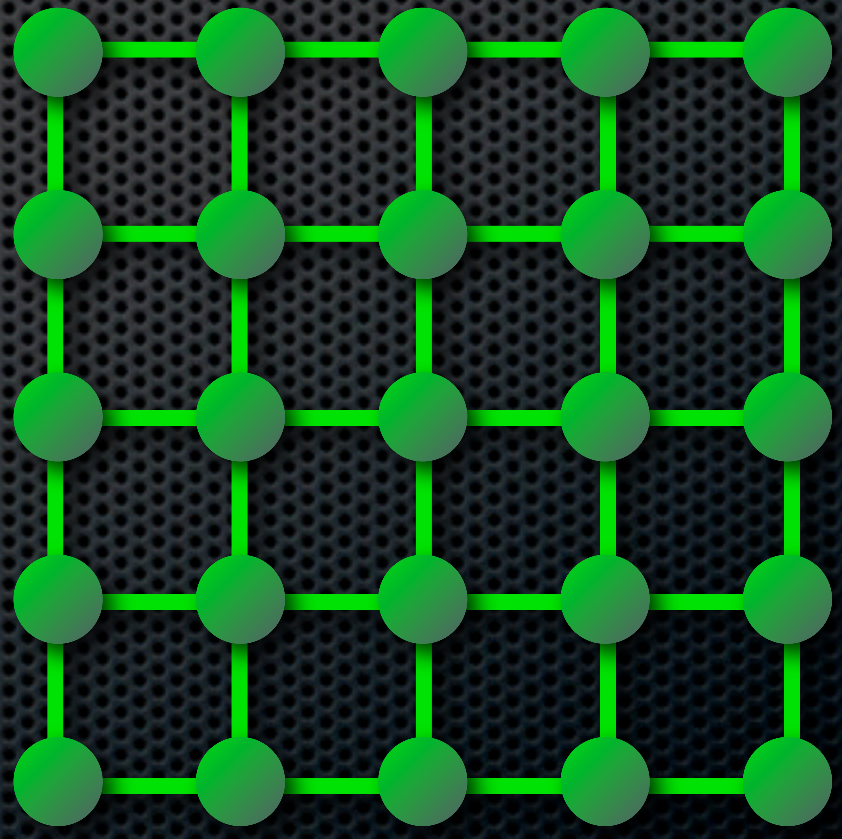

$|\psi_0\rangle$

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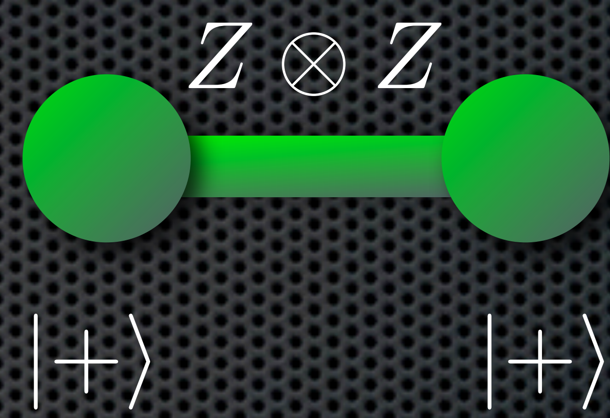

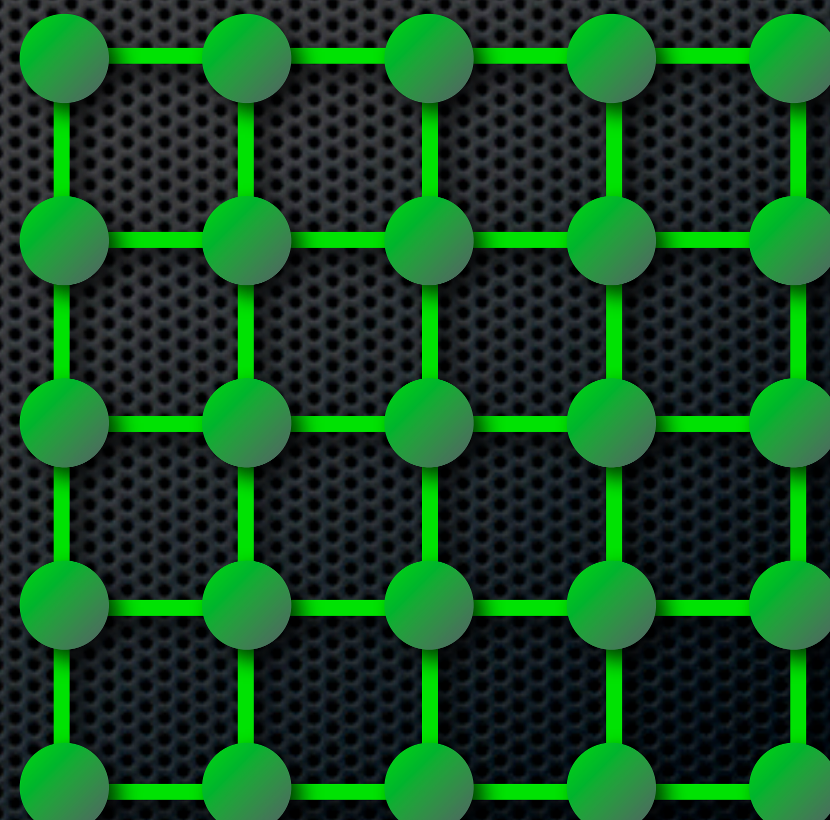

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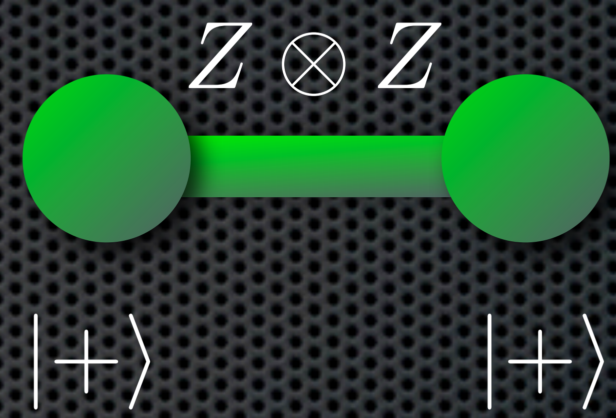

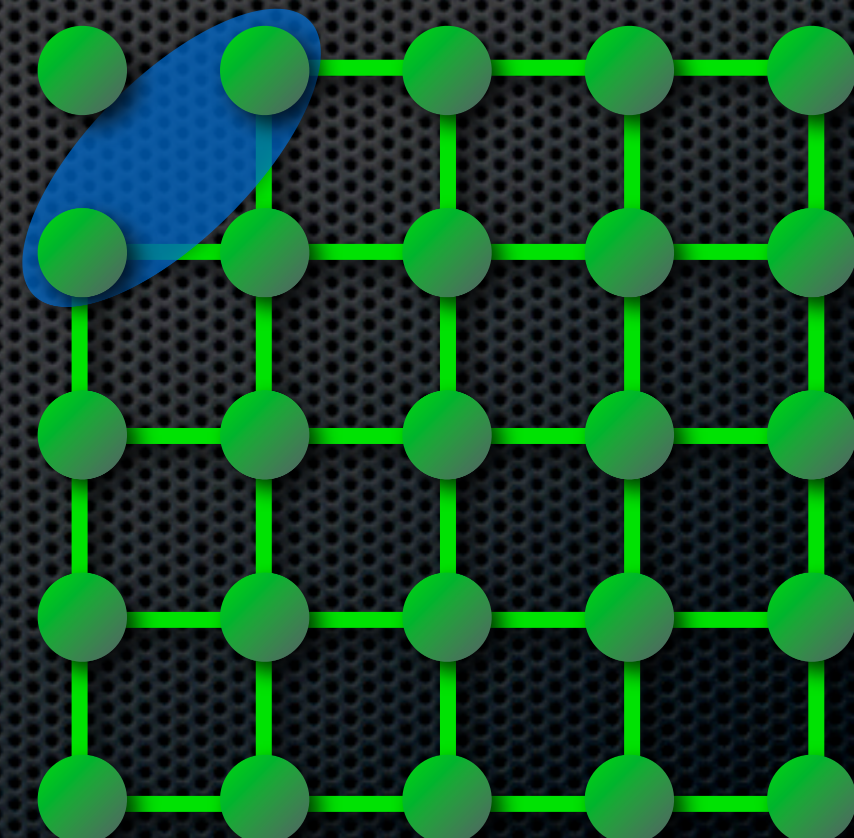

$$|\psi_1\rangle$$

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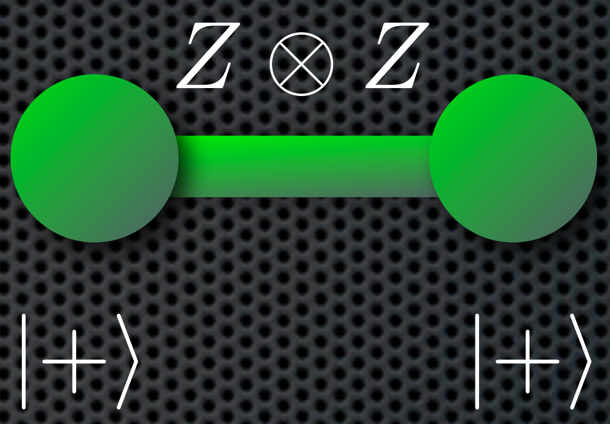

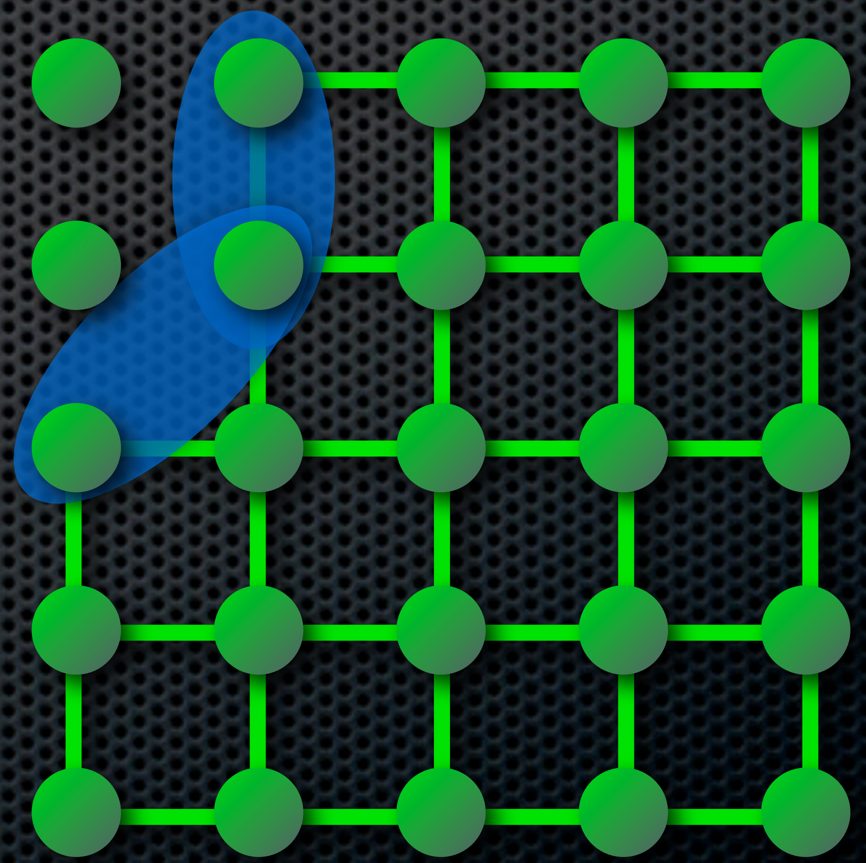

$|\psi_2\rangle$

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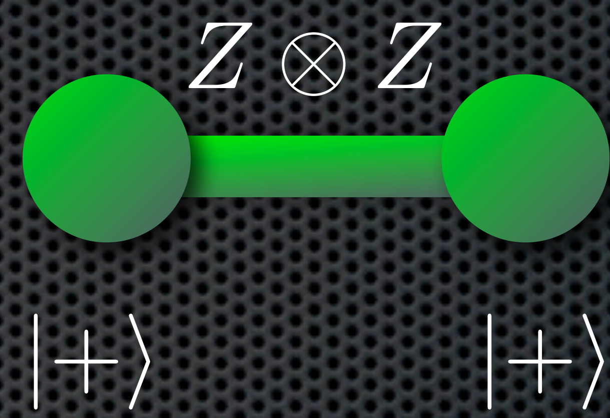

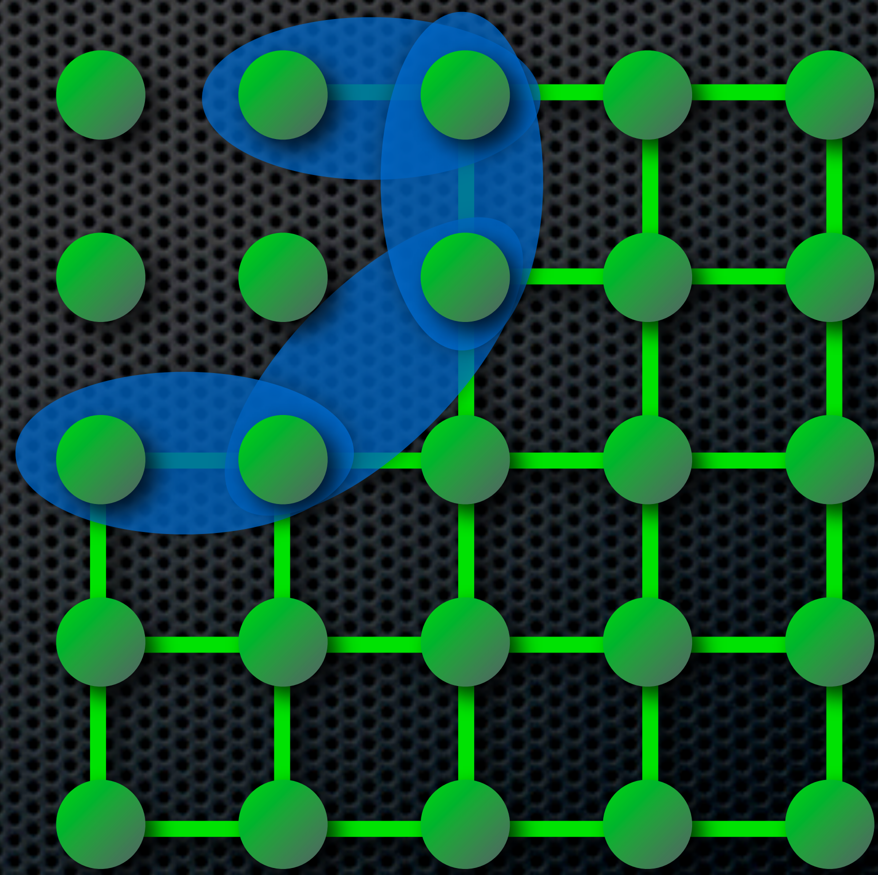

$$|\psi_3\rangle$$

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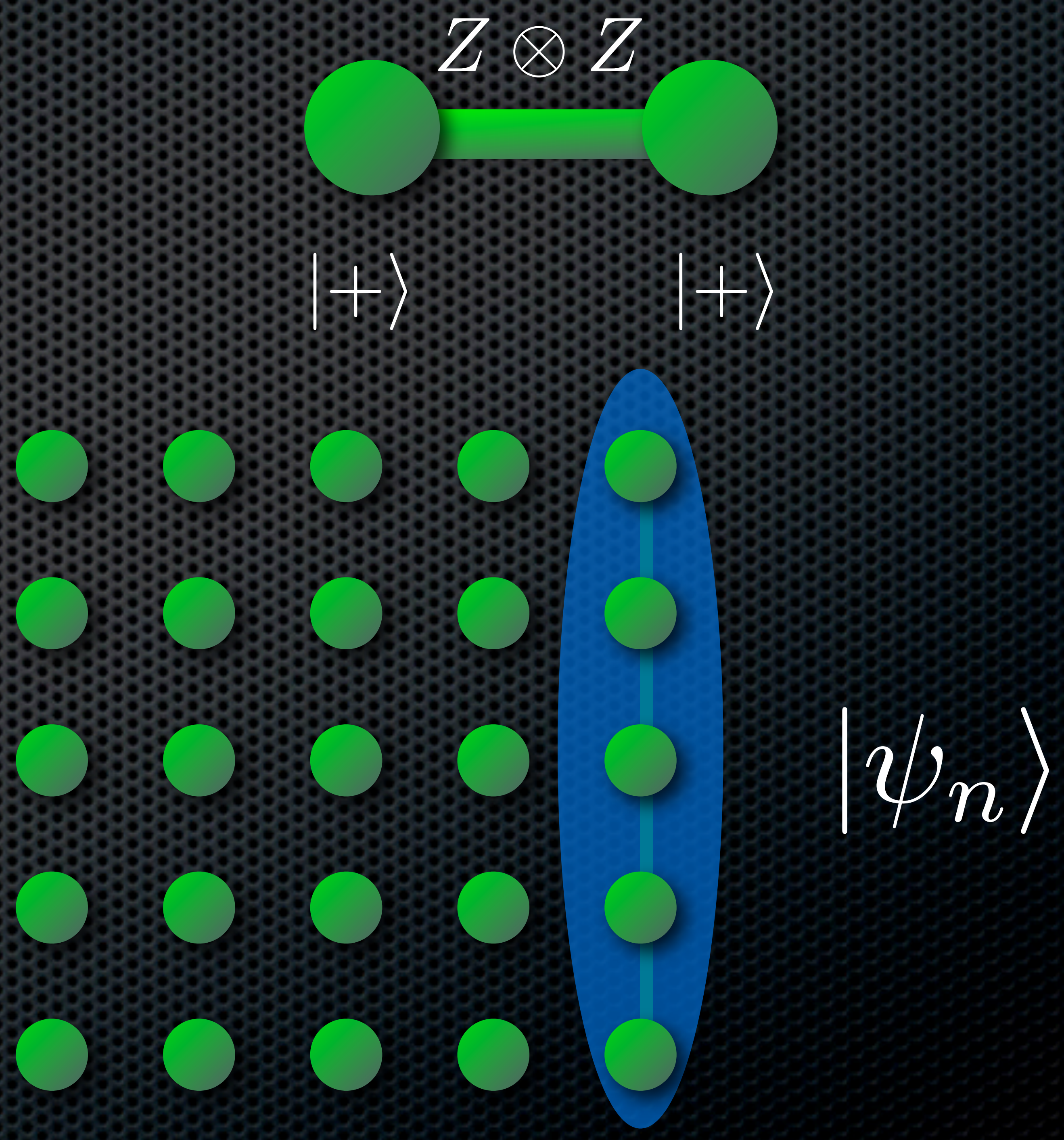

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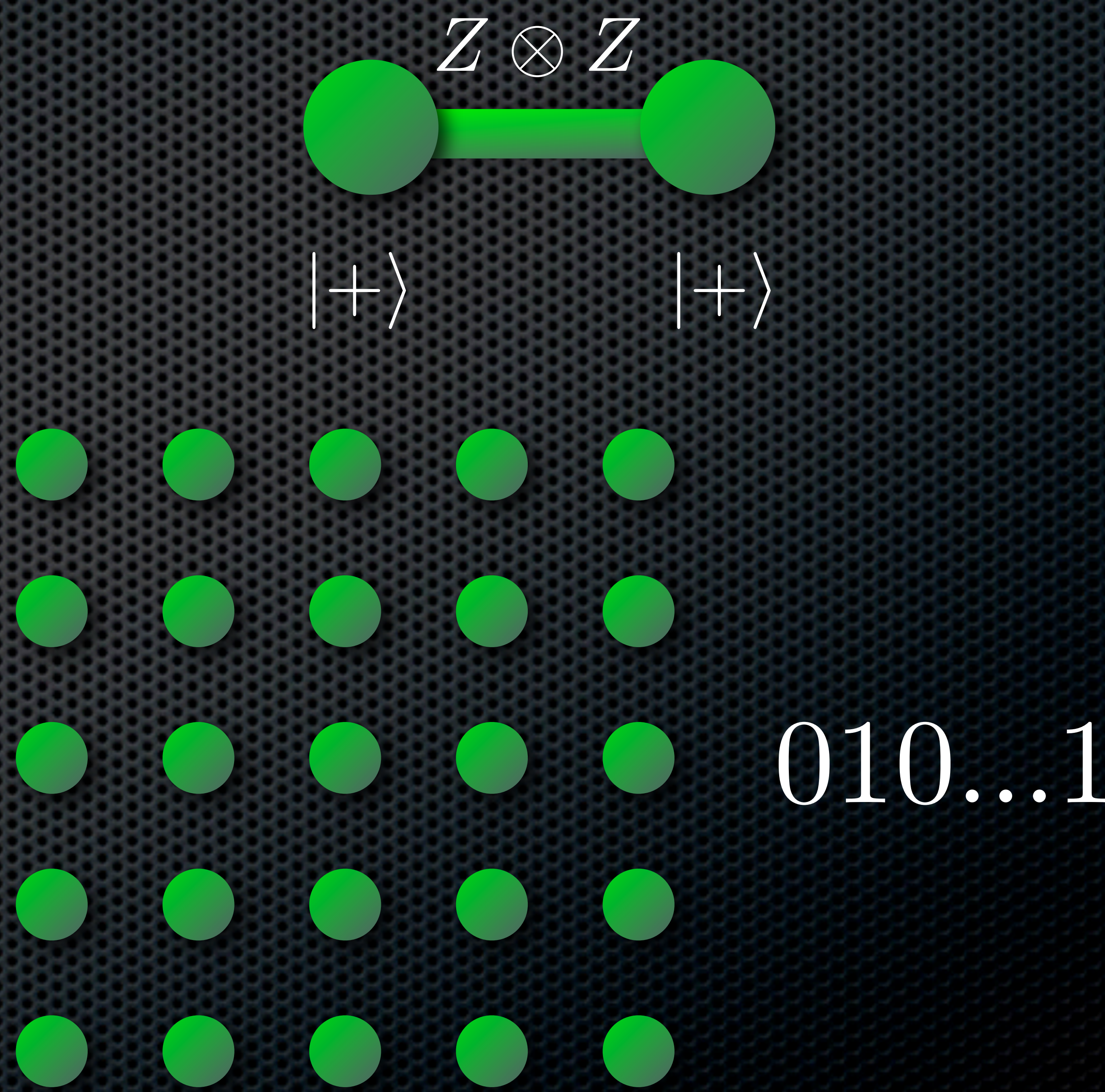

Raussendorf & Briegel PRL 2001

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- Without initial entanglement, it’s clear you can’t do better than BPP.

Universality and entanglement

Question:

What are the necessary and sufficient conditions for a family of n qubit quantum states to be universal for MBQC?

Universality and entanglement

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Necessary conditions:

van den Nest, Miyake, Dür, Briegel 2006

find entanglement measures that must grow “quickly” with n.

Universality and entanglement

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What are the necessary and sufficient conditions for a family of n qubit quantum states to be universal for MBQC?

Sufficient conditions:

Gross, Eisert, Schuch, Pérez-García 2007

find states with special structure in the many-body correlations.

Brennen & Miyake 2008, Doherty & Bartlett 2008

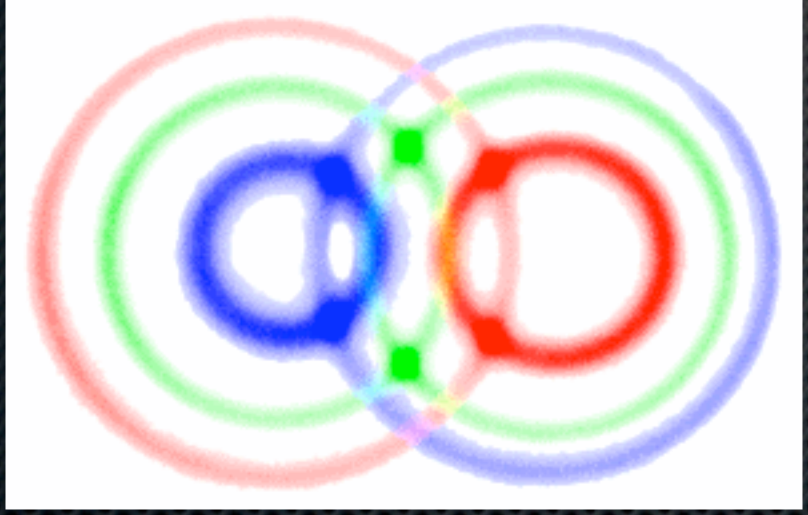
find *ground states* with special structure.

Bridging the divide

Quantum world	Classical world

Bridging the divide

Quantum world	Classical world
Entanglement and correlations	



Bridging the divide

Quantum world

Entanglement
and
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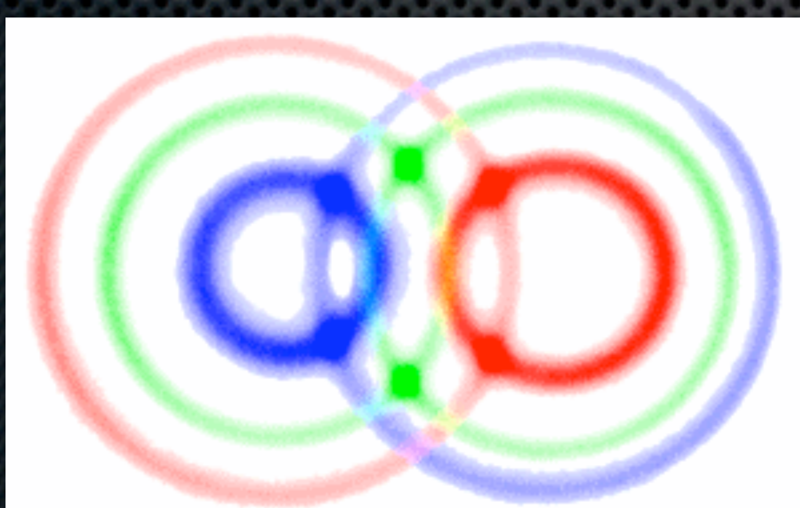
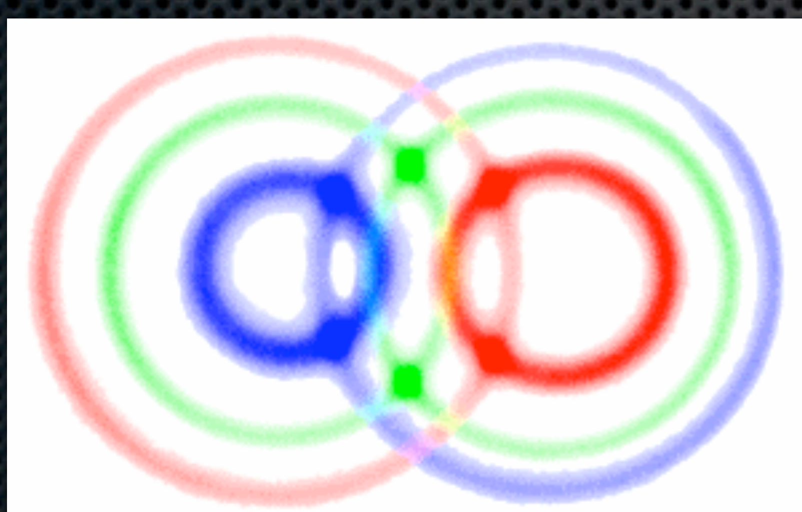


Classical world

Local bases,
Limited processing
power.

Bridging the divide

Quantum world	Classical world
Entanglement and correlations	Local bases, Limited processing power.



MBQC

Local bases, geometric measure

$$E_g(\Psi) = -\log_2 \sup_{\alpha \in \mathcal{P}} |\langle \alpha | \Psi \rangle|^2$$

the set of product states

Answers the question:
How far is the nearest
collection of local bases
$\alpha_1, \alpha_2, \ldots, \alpha_n$?

Large geometric measure

⇕

Far from all product states

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Far from all product states

Theorem 1 (GFE): n qubit states with $E_g > n - O(\log n)$ are useless for MBQC.

Local bases, geometric measure

For concreteness, a state is useless if it fails to provide a polynomial-time MBQC algorithm for Factoring.

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Local bases, geometric measure

For concreteness, a state is useless if it fails to provide a polynomial-time MBQC algorithm for Factoring.

Proof strategy: replace ψ with a classical coin and show there exists a classical algorithm that factors just as well (within poly factors).

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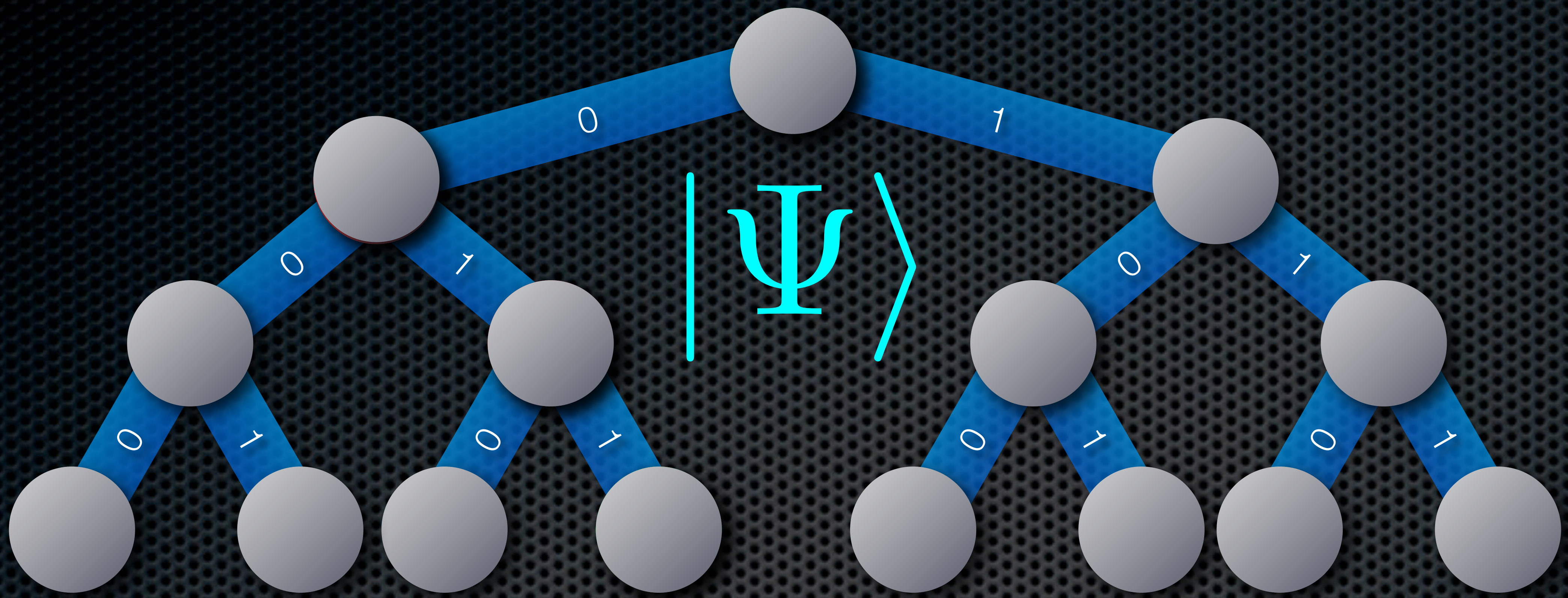
0
1
0
1
0
1
0
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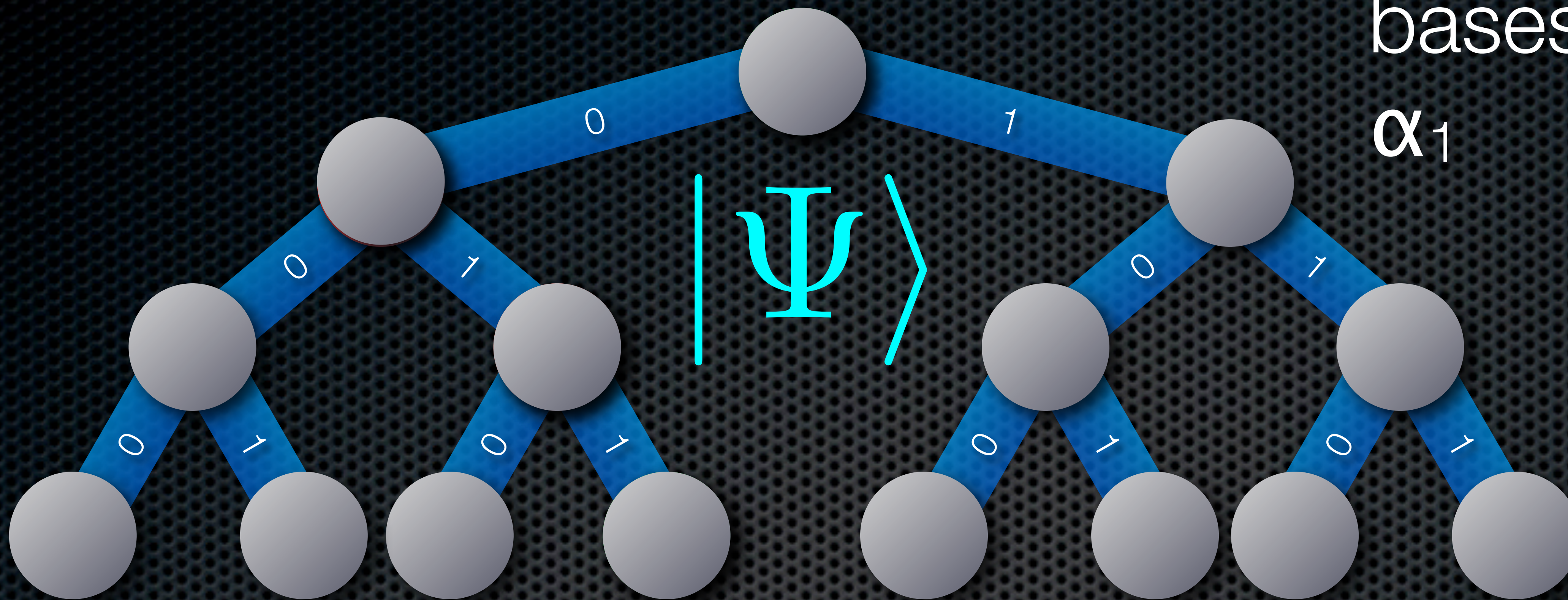
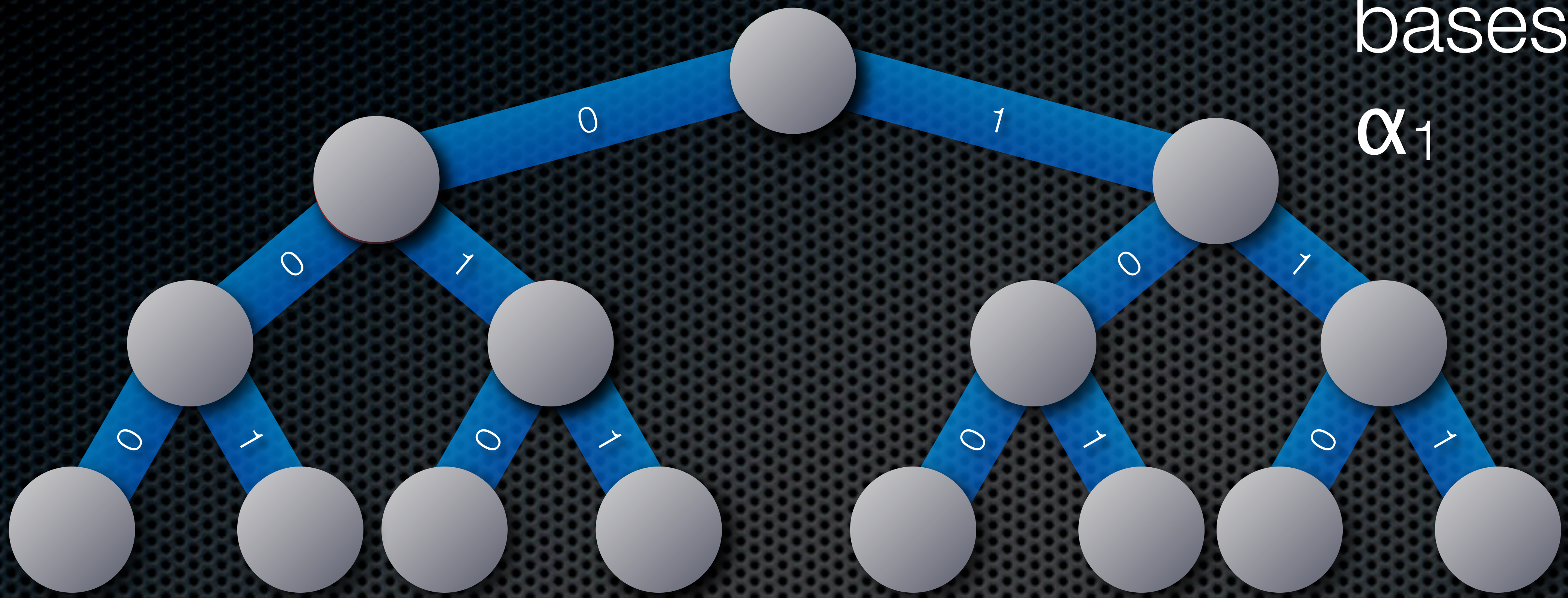
1
0
1
0
1
$|\Psi\rangle$

bases:

α_1





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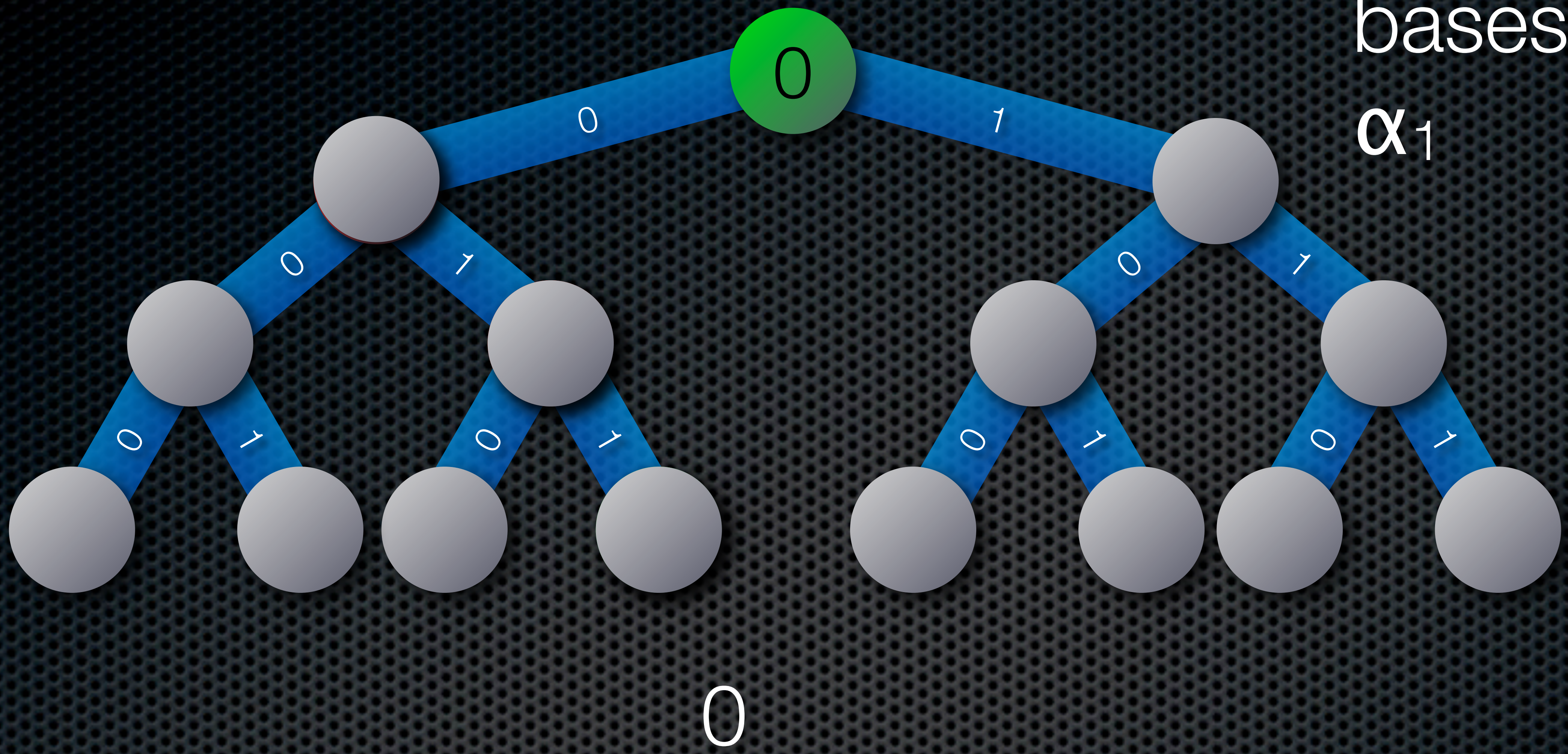
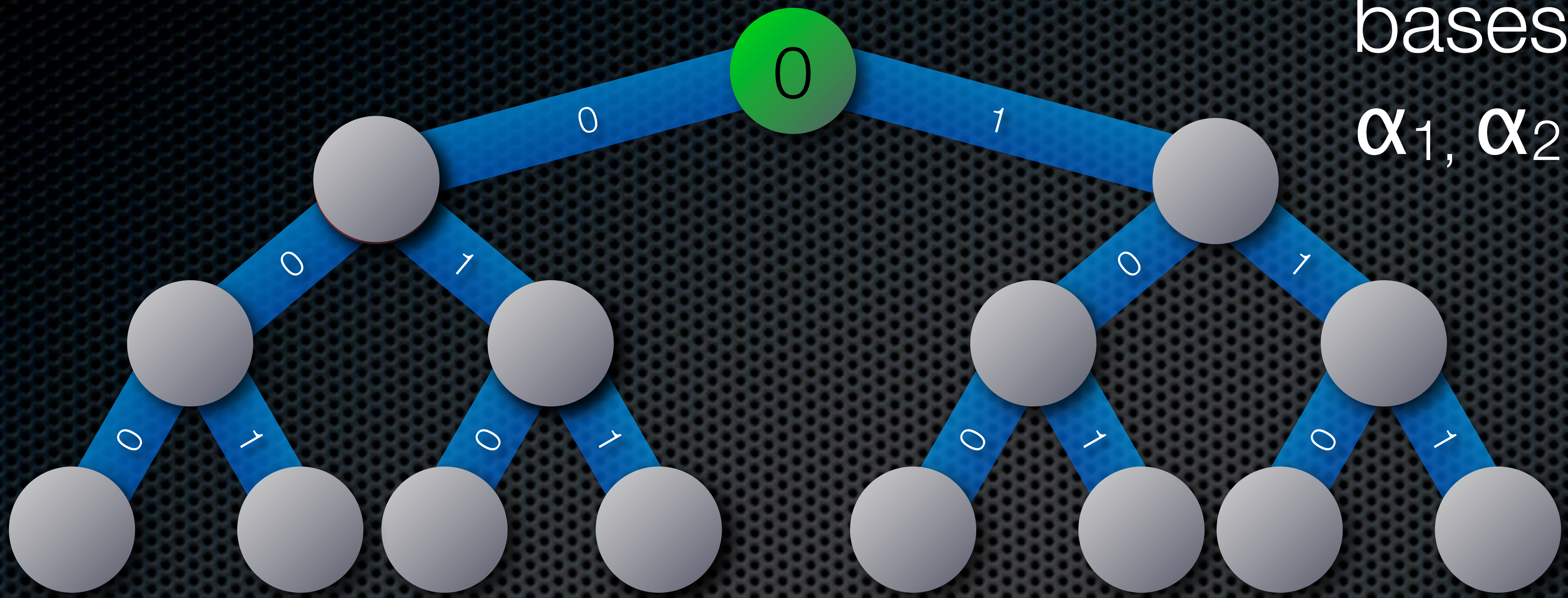
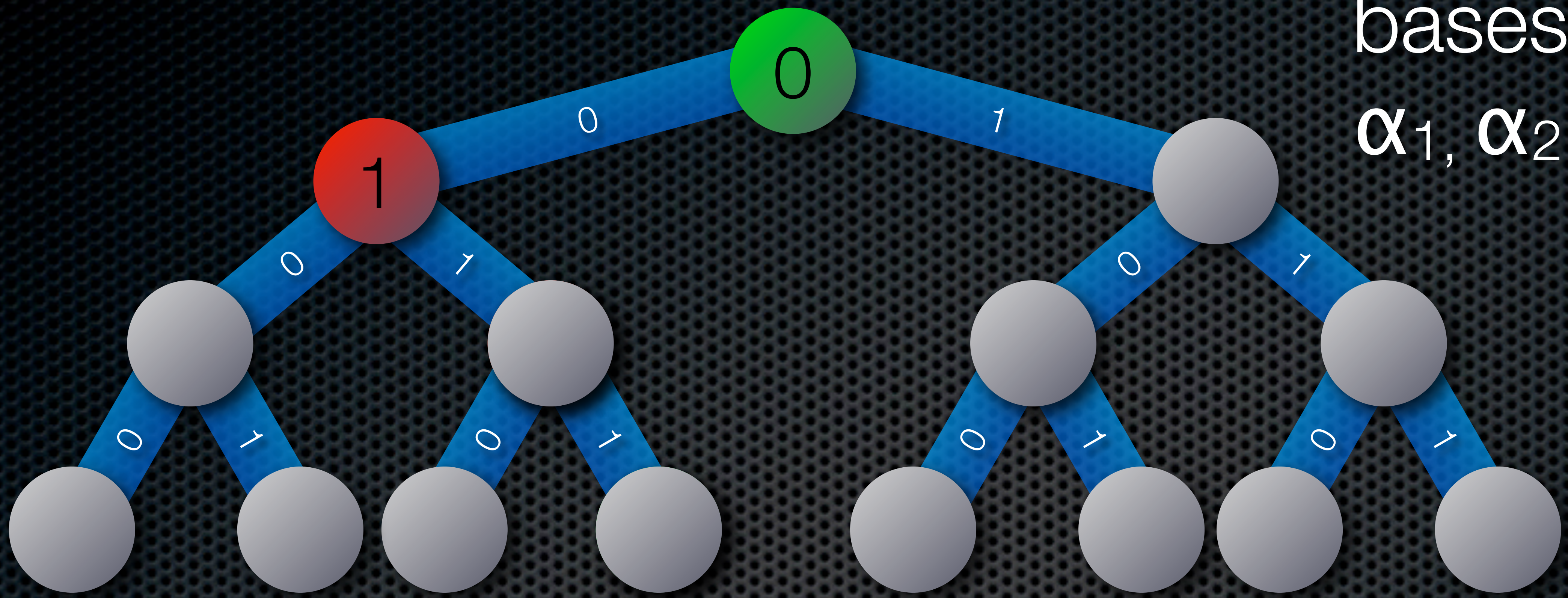




bases:

α_1, α_2

0



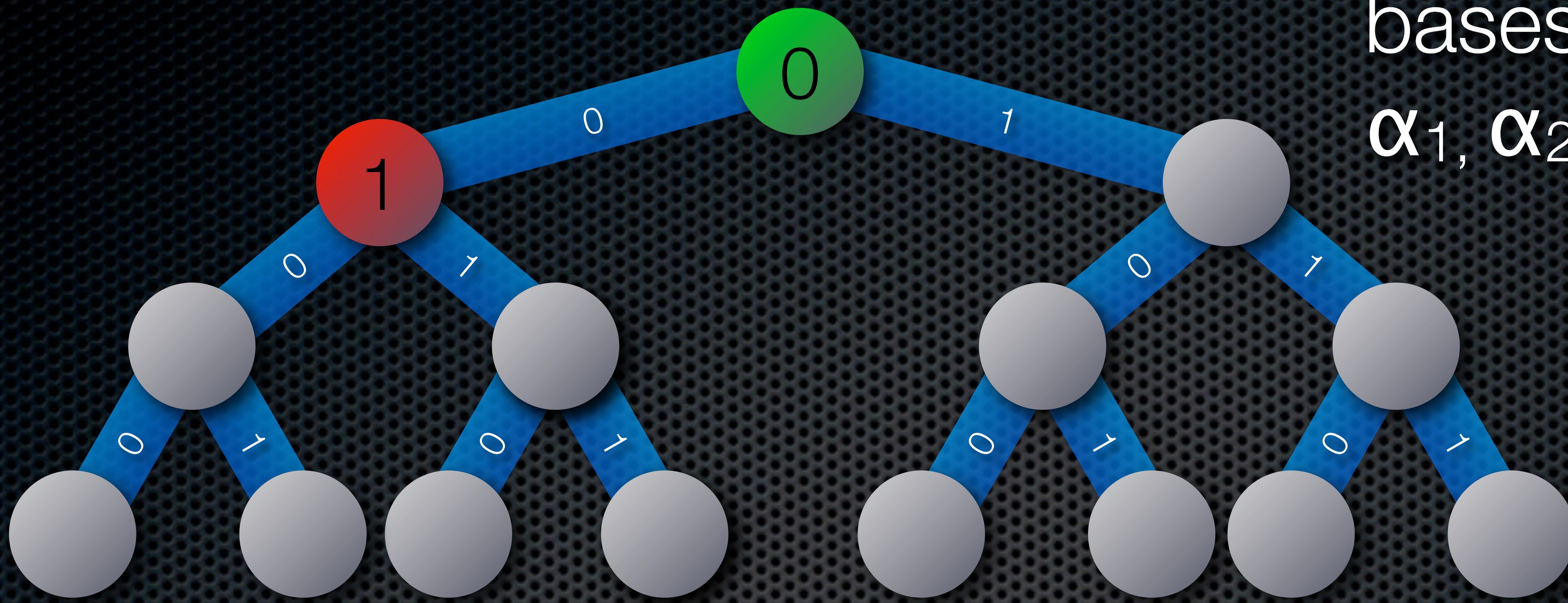
bases:

α_1, α_2

01

bases:

$\alpha_1, \alpha_2, \alpha_3$



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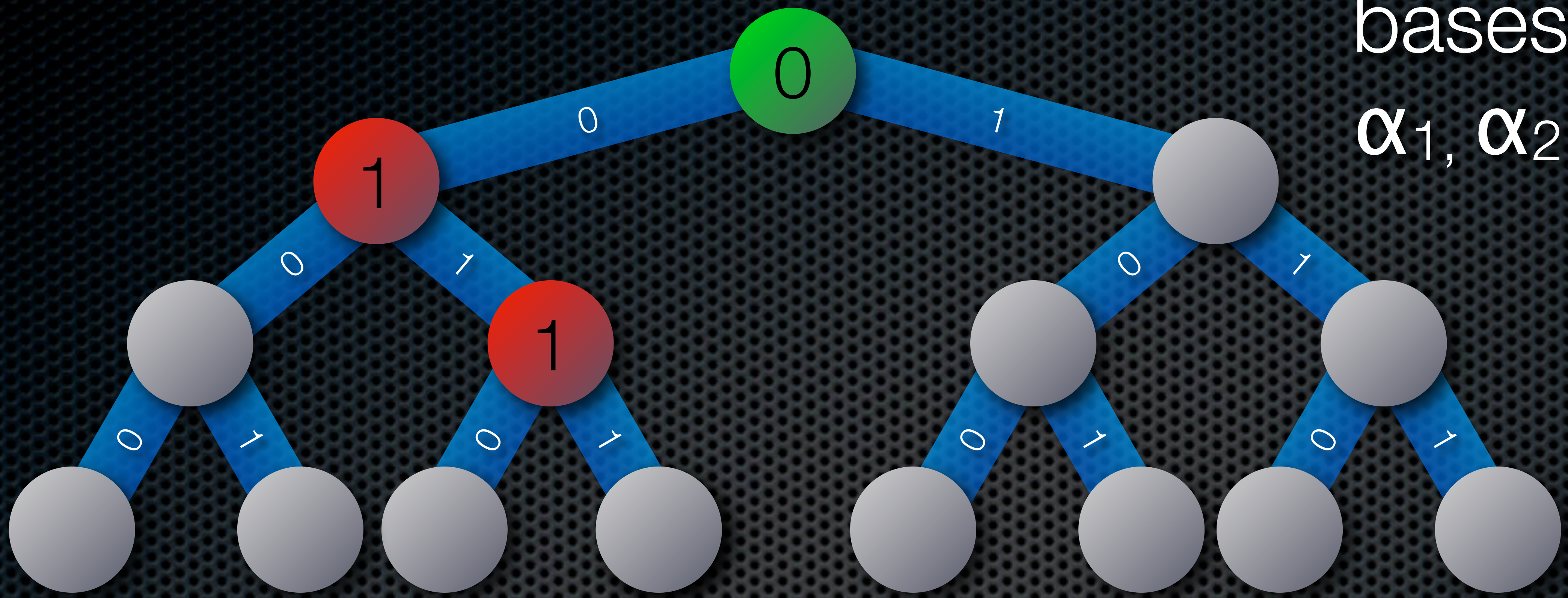


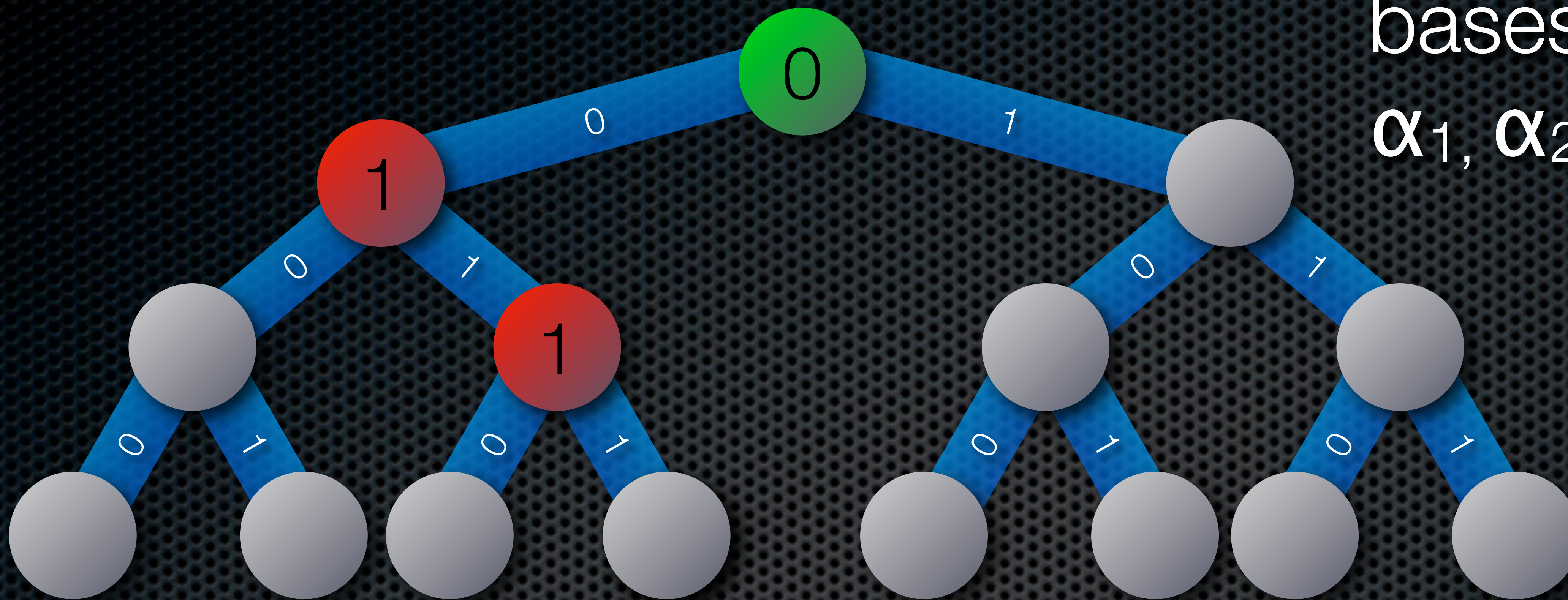
bases:

$\alpha_1, \alpha_2, \alpha_3$

011

bases:

$\alpha_1, \alpha_2, \alpha_3, \alpha_4$



011

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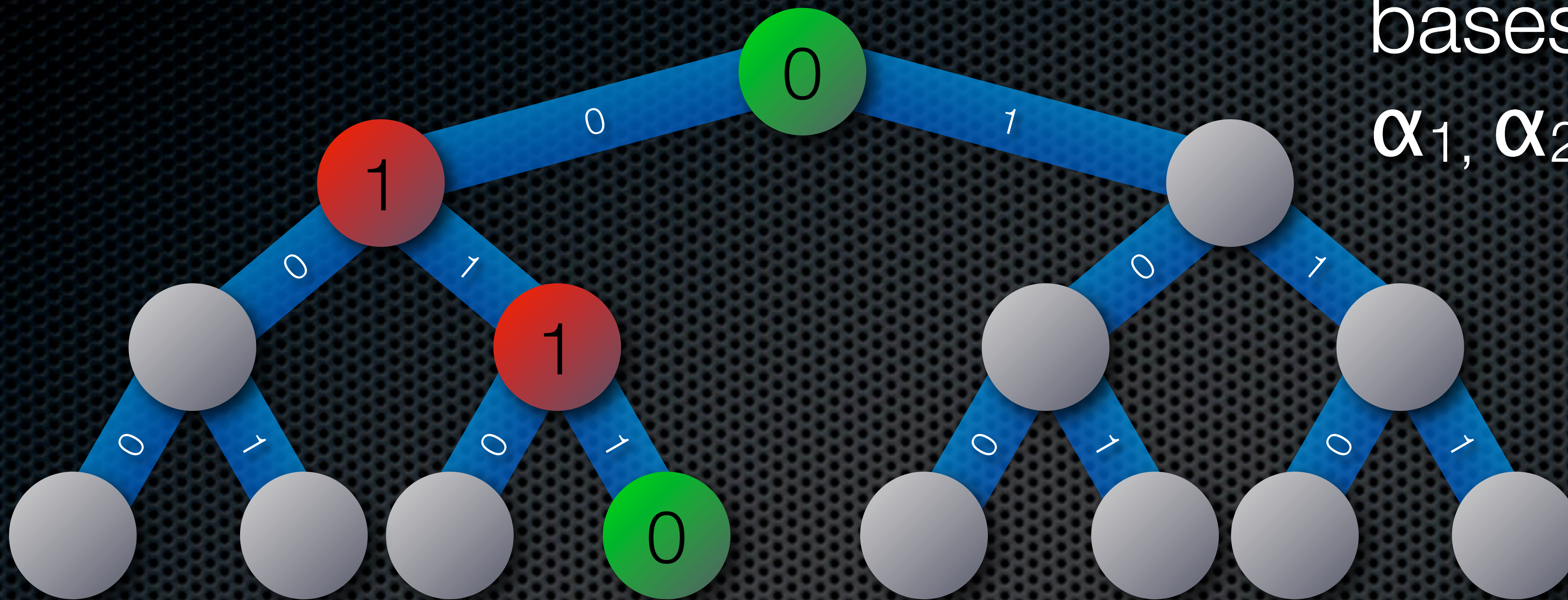
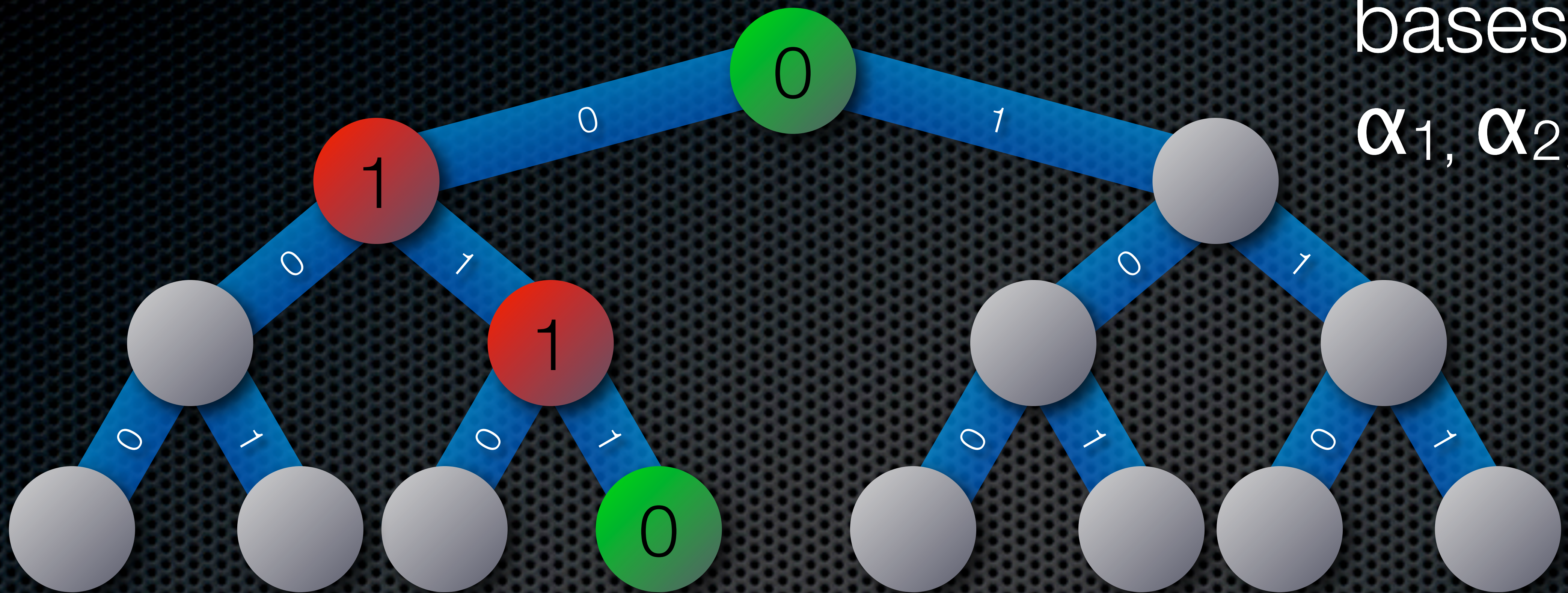


0110



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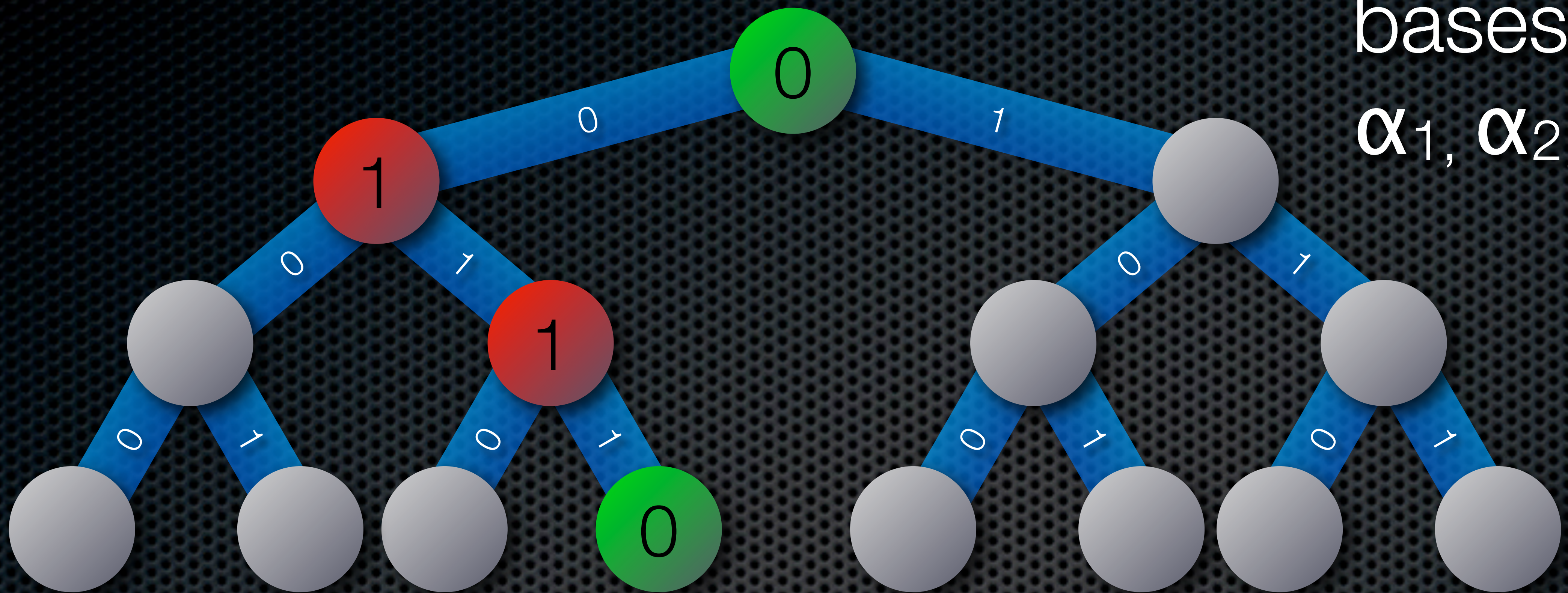
The “good” outcomes G cause the classical control computer to output a valid factorization. We want this to succeed with constant probability, say p>.5



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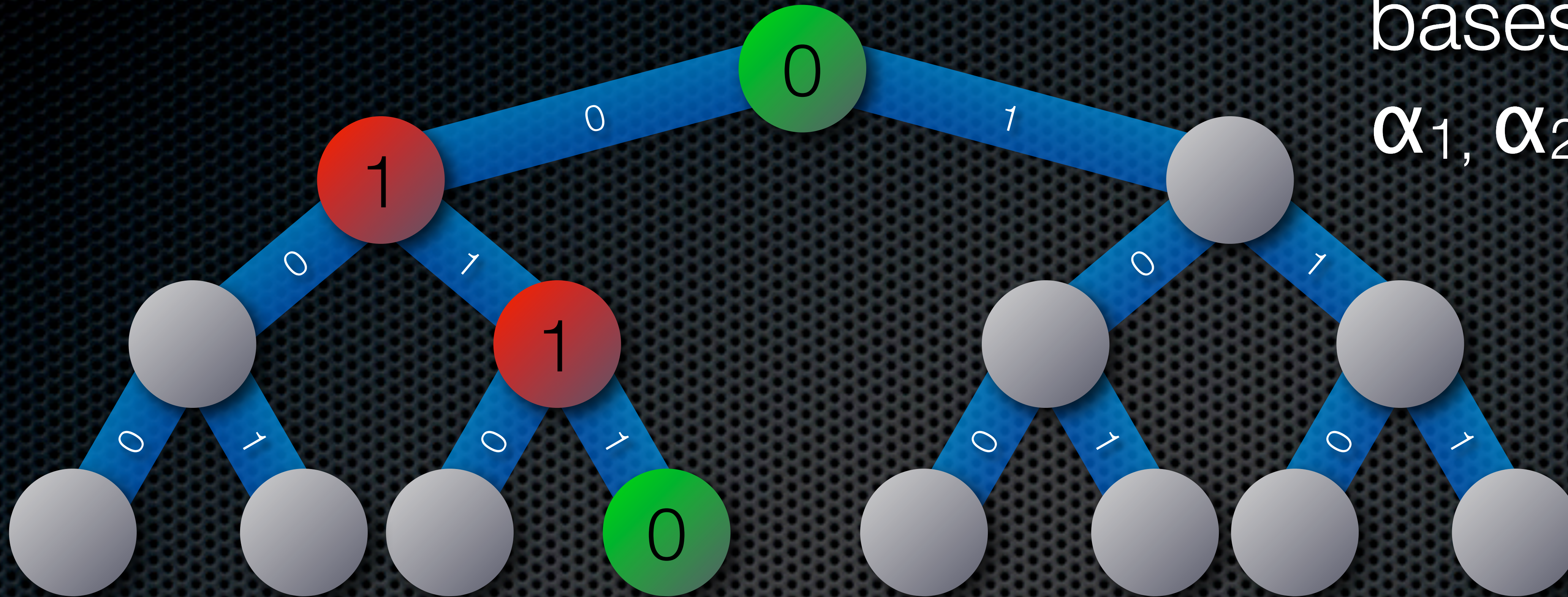

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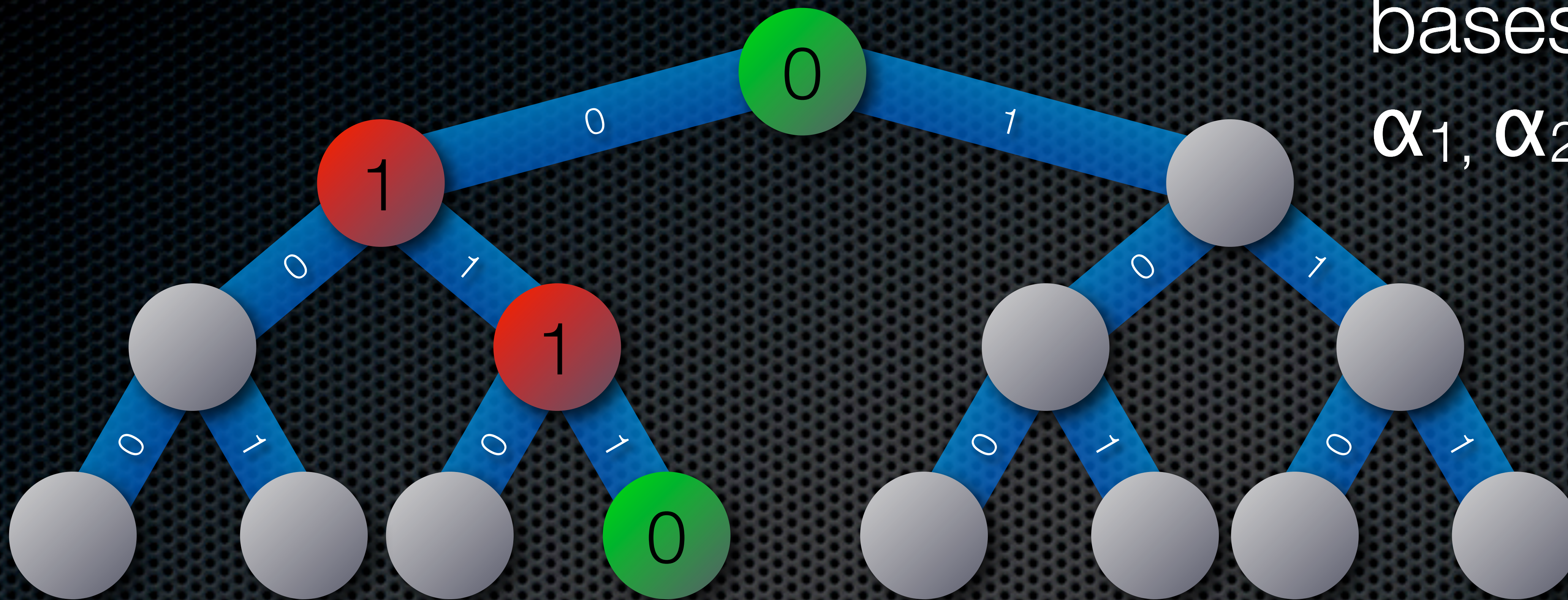
$$|\langle\alpha|\Psi\rangle|^2 \leq 2^{-E_g} \leq 2^{-n+\delta}$$

$$\Rightarrow \frac{|G|}{2^n} > 2^{-\delta-1} = \text{poly}(1/n).$$



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To simulate classically, just ignore the measurement results and use a classical coin!

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In fact, they are abundant.

Theorem 2 (GFE): The fraction of n qubit states with

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Theorem 2 (GFE): The fraction of n qubit states with $E_g < n - O(\log n)$ is less than $\exp(-n^2)$.

The proof involves standard measure concentration arguments (via ϵ-nets) and known results about random states

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Can provably useless states be created efficiently?

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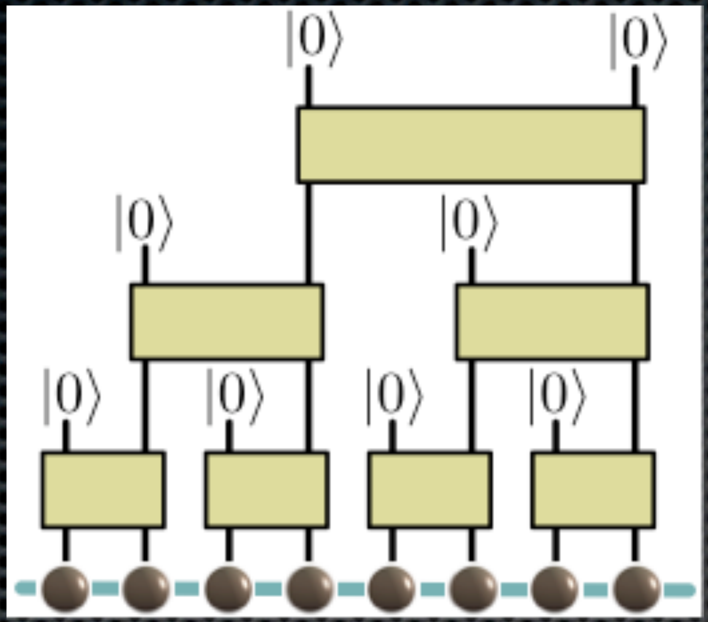

We can get to $E_g > n - o(n)$ using a TTN construction.

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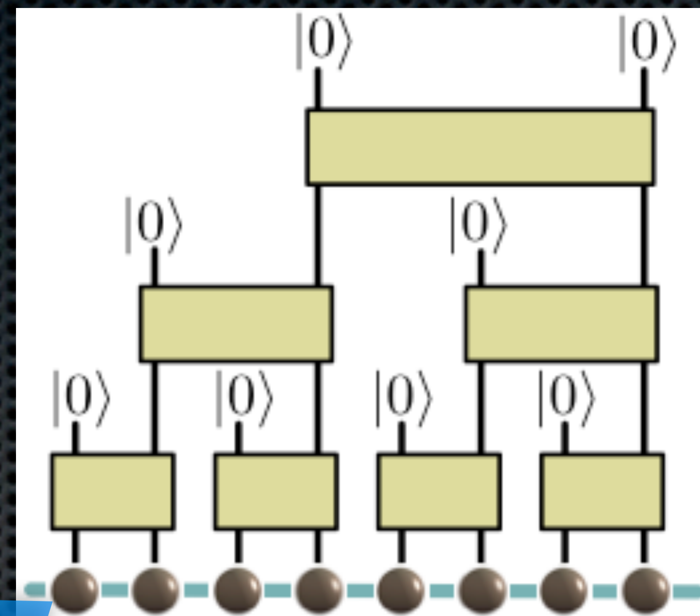

d-level systems

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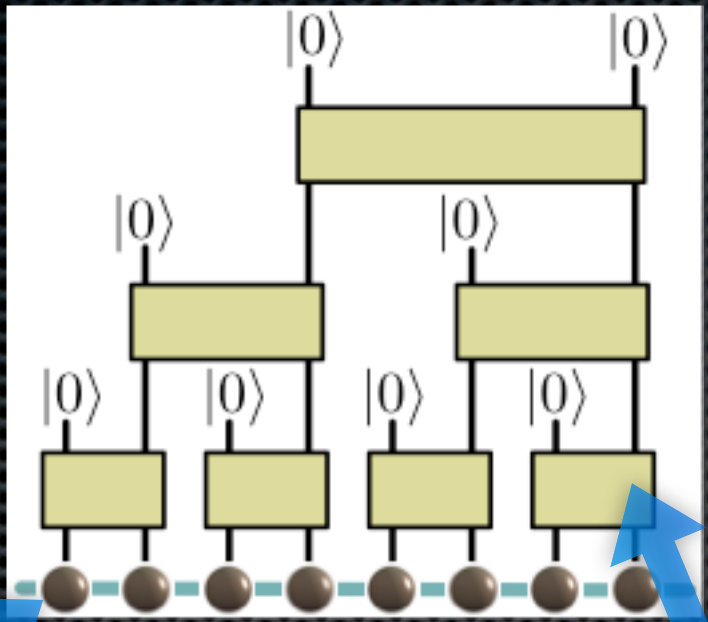

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Isometry V = V(U)

$$V : \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$$

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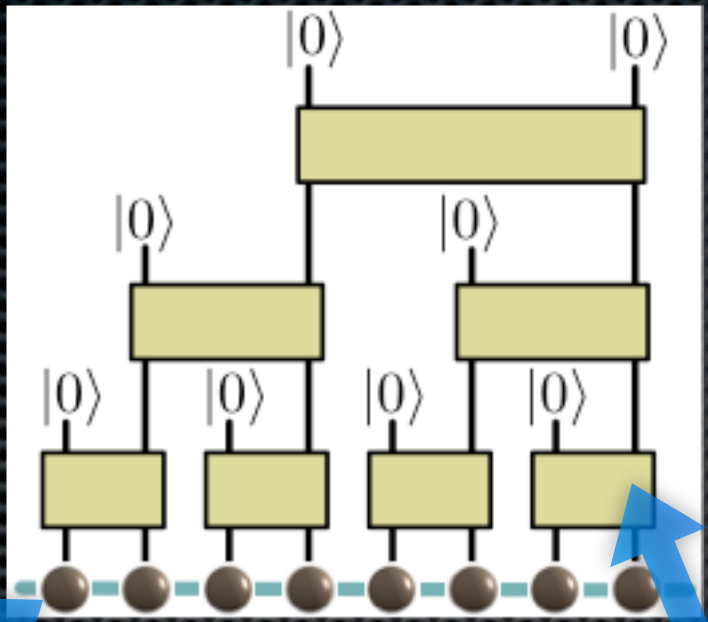

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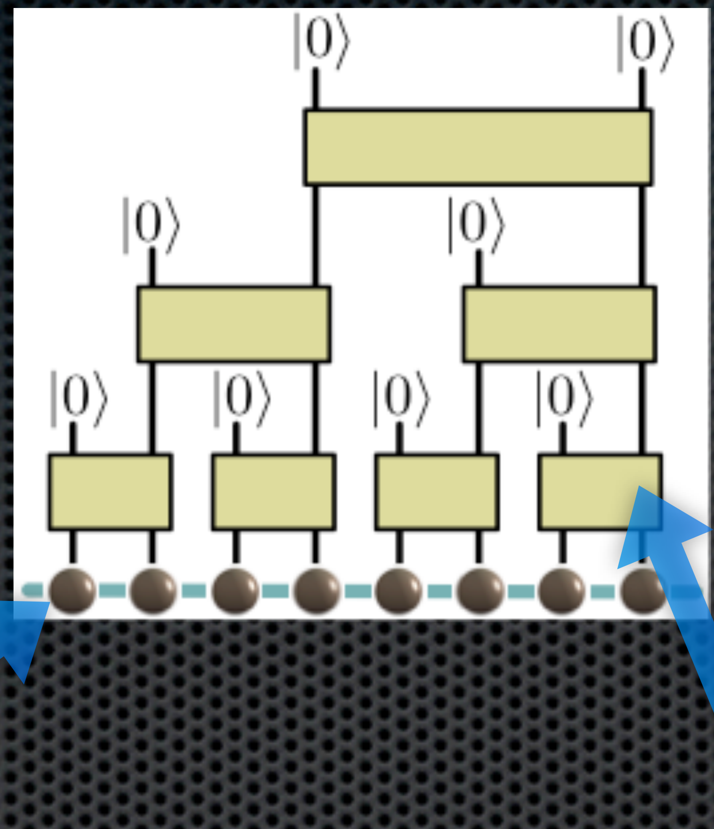

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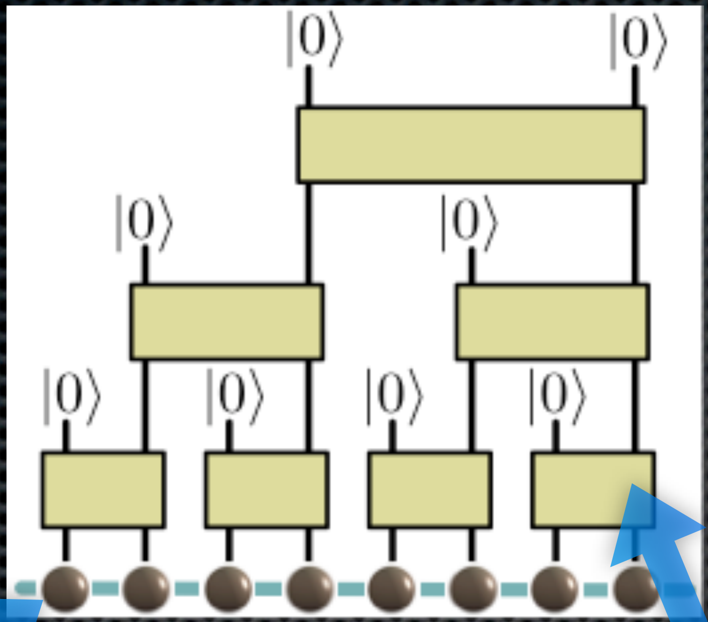

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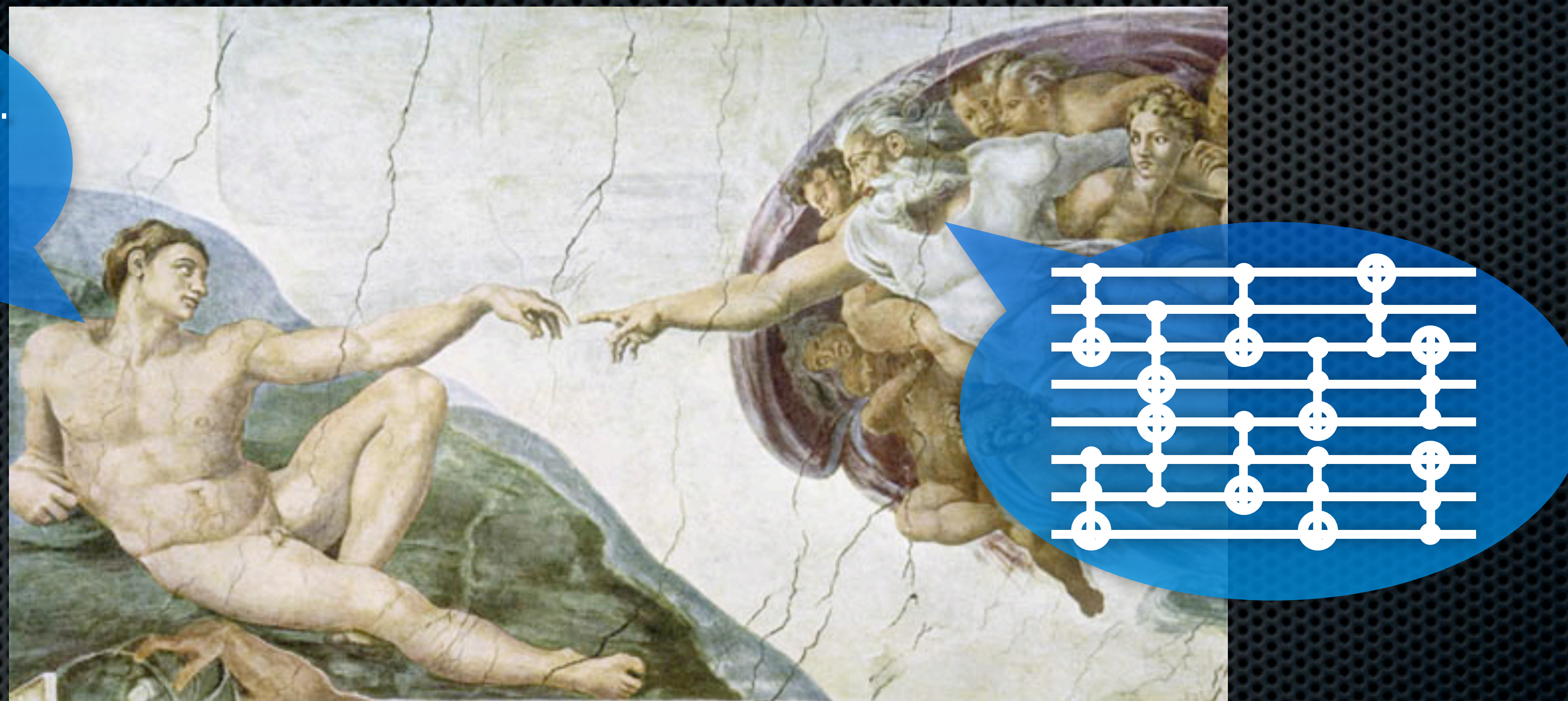
$E_g > n - o(n)$

Decision problems

I have a generic Ψ.
Can I compute
anything with it?

$$|\Psi\rangle$$



- For almost every state Ψ, there is no poly-bounded classical control circuit which allows a significant advantage over classical randomness.
  Only problems in BPP can be solved. (BMW '08)

$$\Pr_{\Psi}\left\{\exists C \ \left|C(\Psi) - C(2^{-n}\mathbb{1})\right| > \epsilon\right\} \leq \left(8^8 w\right)^{3v} e^{-c\epsilon^2 2^n}$$

Randomness vs entanglement?

Random states such that $E_g \leq \log K + O(1)$ also offer no advantage!

- Choose nK states at random from $\mathbb{C}^2$ to construct the following (where K is superpolynomial in n):

$$R := \sum_{j=1}^{K} |\psi_j^{(1)}\rangle\langle\psi_j^{(1)}| \otimes \cdots \otimes |\psi_j^{(n)}\rangle\langle\psi_j^{(n)}|$$

- Randomly pick a state from the support of R then:

$$|\Psi\rangle = \frac{1}{\sqrt{\langle\Psi_0|R|\Psi_0\rangle}} \sqrt{R}|\Psi_0\rangle$$

$$\Pr_\Psi \left\{\exists C \; \left|C(\Psi) - C(2^{-n}\mathbb{1})\right| > \epsilon\right\} \leq \left(2^n + \left(8^8 w\right)^{3v}\right) e^{-c'\epsilon^2 K^{1/3}}$$

Questions

- Can we derandomize these constructions?
- Can Hastings' techniques give improved bounds?
- Are efficiently created states subject to this effect?
- What happens with a polynomial number of copies?
- What implications does this have for the circuit model?